First Order Theories - The Theory of Equality

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1 Equality
It is the simplest first-order theory.
Basics

It is the simplest first-order theory. Its signature is:

$$\Sigma_E = \{=, a, b, c, \ldots, f, g, h, \ldots p, q, r, \ldots\}.$$
The Theory of Equality

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(A2.) \((\forall x)(\forall y)\ (x = y) \rightarrow (y = x)\).
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\. (A1.) (\( \forall x \)) \( x = x \).
\. (A2.) (\( \forall x)(\forall y \)) (\( x = y \)) \( \rightarrow \) (\( y = x \)).
\. (A3.) (\( \forall x)(\forall y)(\forall z \)) (\( x = y \)) \( \land \) (\( y = z \)) \( \rightarrow \) (\( x = z \)).
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(A3.) $$(\forall x)(\forall y)(\forall z) \ (x = y) \land (y = z) \rightarrow (x = z).$$  
(A4.) $$(\forall \overline{x})(\forall \overline{y}) \ (\land_{i=1}^{n} (x_i = y_i)) \rightarrow [f(\overline{x}) = f(\overline{y})].$$
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(A4.) \((\forall \vec{x})(\forall \vec{y}) \, (\land_{i=1}^{n} (x_i = y_i)) \rightarrow [f(\vec{x}) = f(\vec{y})]\).
(A5.) \((\forall \vec{x})(\forall \vec{y}) \, (\land_{i=1}^{n} (x_i = y_i)) \rightarrow [p(\vec{x}) = p(\vec{y})]\).

In (A4.) and (A5.), \(\vec{x} = (x_1, x_2, \ldots x_n)\), and \(\vec{y} = (y_1, y_2, \ldots y_n)\).
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Note

(i) A **axiom schema** stands for a set of axioms, each an instantiation of the parameters.
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Note

(i) A axiom schema stands for a set of axioms, each an instantiation of the parameters.

(ii) The theory of equality is undecidable.
Equality

Example

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First Order Theories
Example

Is the formula $F : (a = b) \land (b = c) \rightarrow g(f(a), b) = g(f(c), a)$ valid?