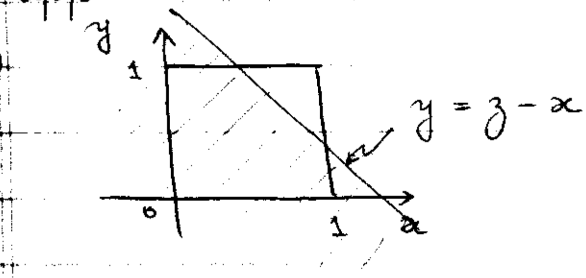
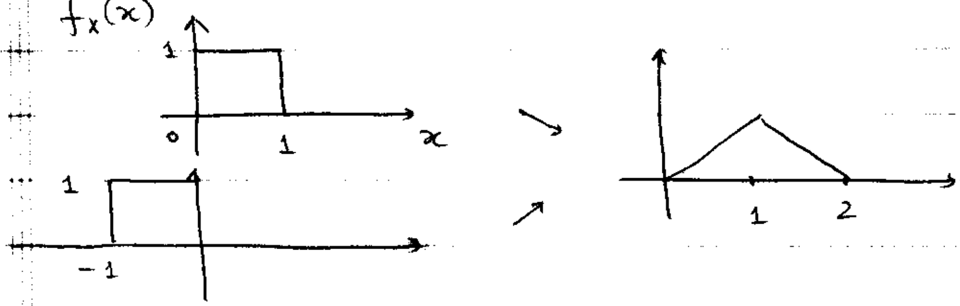
(15). The density of Z can be evaluated using 2 approaches.

(a).

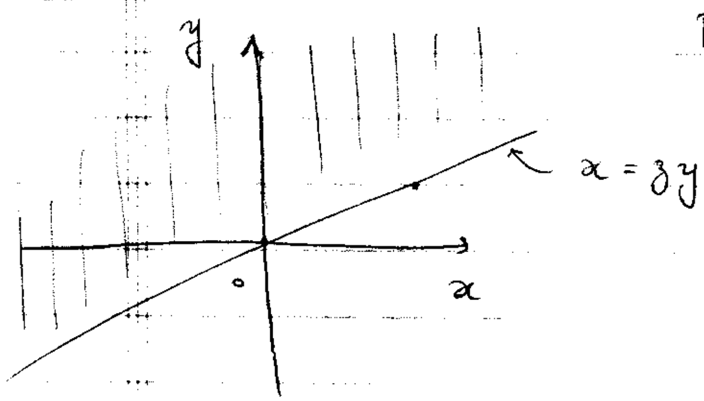


Integrate the region below $x + y \le z$ and differentiate.

(b). Convolve two densities:



ex. 2 : Let $Z = X/Y$



$$\Pr[Z \le z] = \Pr[X/Y \le z]$$

$x = zy$

$$= \int_{-\infty}^{+\infty}\int_{x/z}^{+\infty} f_{X,Y}(x,y)\,dx\,dy$$

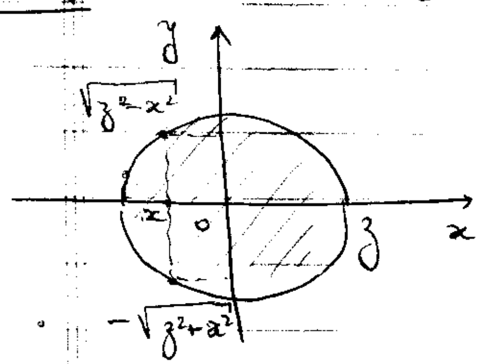
"x" "y"

$(z > 0)$

And the density of Z is

$$f_Z(z) = \frac{1}{z^2}\int_{-\infty}^{+\infty} f_{X,Y}\left(x, \frac{x}{z}\right)\cancel{x}\,dx$$

ex. 3 : Let $Z = \sqrt{X^2 + Y^2}$



$$P_Z(z) = \Pr[\sqrt{X^2+Y^2} \le z] = \Pr[X^2 + Y^2 \le z^2]$$

$$= \int_{-z}^{z}\int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2+x^2}} f_{X,Y}(x,y)\,dy\,dx$$

(16).

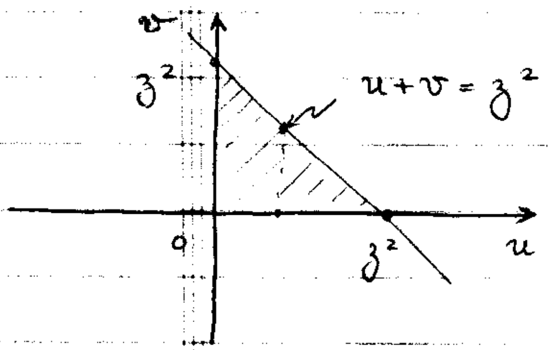
Let $X \sim N(0,1)$,
$Y \sim N(0,1)$, $X \perp\!\!\!\perp Y$.
$\Rightarrow \boxed{U \perp\!\!\!\perp V}$

A solution:

Let $U = X^2$ and
$V = Y^2$, then
$Pr\{U + V \le z^2\}$

Recall,

$$P_U(u) = Pr[U \le u]$$
$$= Pr[X^2 \le u] = Pr[-\sqrt{u} \le X \le \sqrt{u}]$$
$$= \int_{-\sqrt{u}}^{\sqrt{u}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$
$$f_U(u) = \frac{1}{\sqrt{2\pi u}} e^{-u/2} \cdot I_{\{u \ge 0\}}$$ ← (χ^2 with one degree of freedom)



$$Pr\{U + V \le z^2\}$$
$$= \int_{0}^{z^2} \int_{0}^{z^2 - u} \underbrace{f_{U,V}(u,v)}_{f_U(u) \cdot f_V(v)} \, du \, dv \qquad U \perp\!\!\!\perp V$$
"u" "v"

Hard! → $$= \int_0^{z^2} \frac{1}{\sqrt{2\pi u}} e^{-u/2} \cdot \int_0^{z^2 - u} \frac{1}{\sqrt{2\pi v}} e^{-v/2} \, dv \, du$$

Statement: Sum of independent central χ^2 RV's is a χ^2 RV with "sum" of degrees of freedom.

Thus,

$$W = X + Y \sim \frac{e^{-w/2}}{2} \cdot I_{\{w \ge 0\}}.$$

Then,

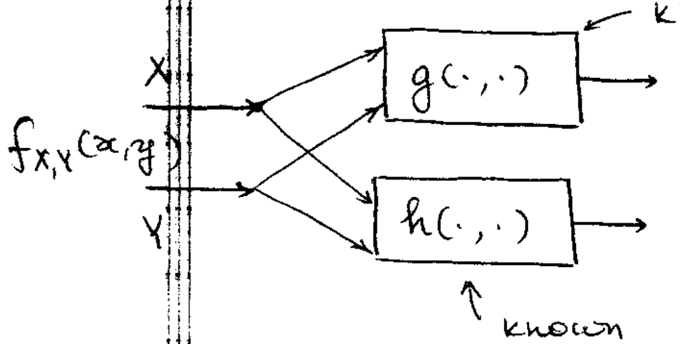
$$P_Z(z) = Pr[W \le z^2] = \frac{1}{2} \int_0^{z^2} e^{-w/2} dw$$

$$f_Z(z) = \frac{1}{2} \cdot 2z \, e^{-z^2/2} \, I_{\{z \ge 0\}}$$
$$= z \cdot e^{-z^2/2} \cdot I_{\{z \ge 0\}}$$ ~ Rayleigh distributed RV.

(17).

Two Functions of Two RVs

Let X and Y be distributed as $f_{X,Y}(x,y)$

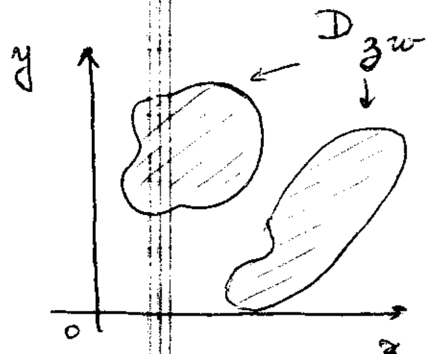


Find: joint cdf and pdf of the pair (Z, W).

1. Find equivalent events

$$D_{zw} = \{(x,y):\ g(x,y) \le z,\ h(x,y) \le w\}$$

$$\Pr[Z \le z,\ W \le w]$$
$$= \Pr[g(X,Y) \le z,\ h(X,Y) \le w]$$
$$= \Pr[(X,Y) \in D_{zw}]$$

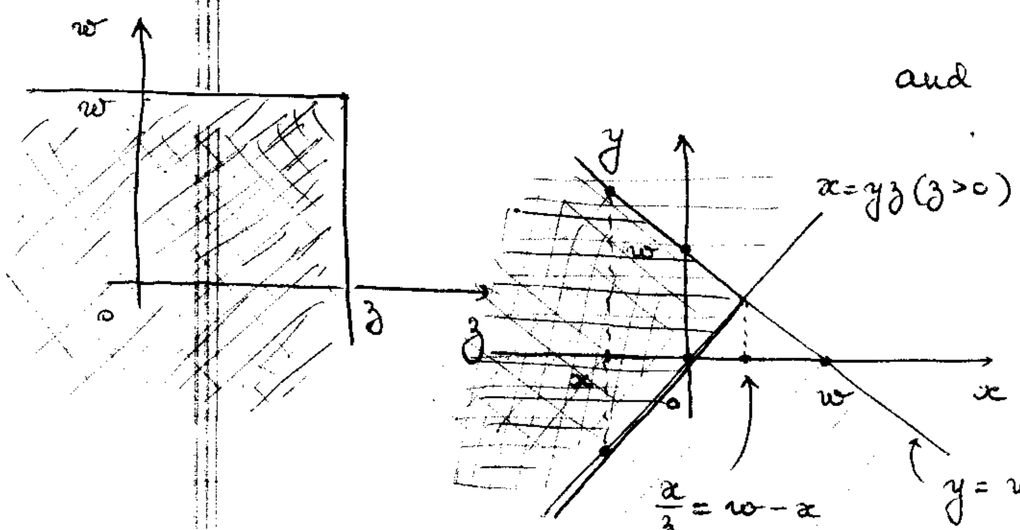


ex. Let $W = X + Y$
$Z = X/Y$

and $f_{X,Y}$ be given.

$$\Pr[Z \le z,\ W \le w]$$
$$= \Pr[X/Y \le z,\ X+Y \le w]$$
$$= \int_{-\infty}^{\frac{wz}{1+z}} \int_{x/z}^{w-x} f_{X,Y}(x,y)\,dy\,dx$$

"x" "y"



$$\frac{x}{z} = w - x$$
$$x = \frac{wz}{1+z}$$

(18).

Joint Density :

Let $Z = g(X, Y)$ and $W = h(X, Y)$

g known, h known.

Let $f_{X,Y}(x, y)$ is given.

Theorem: To find $f_{Z,W}(z, w)$ we solve the system of equations:

$$\begin{cases} g(x, y) = z \\ h(x, y) = w \end{cases} \qquad (1)$$

Let $\{(x_n, y_n)\}$ be real roots of the system.

Then

$$f_{Z,W}(z, w) = \frac{f_{X,Y}(x_1, y_1)}{|J(x_1, y_1)|} + \dots + \frac{f_{X,Y}(x_n, y_n)}{|J(x_n, y_n)|} + \dots$$

where

$$J(x, y) = \det \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} = \frac{1}{\det \begin{bmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{bmatrix}}$$

is the Jacobian of the transformation (1).

ex. (Rotation)

Let
$$Z = X\cos\varphi + Y\sin\varphi$$
$$W = -X\sin\varphi + Y\cos\varphi$$

$$\begin{bmatrix} Z \\ W \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}}_{\text{orthogonal matrix (rotation)}} \cdot \begin{bmatrix} X \\ Y \end{bmatrix}$$

$QQ^T = I$

$Q^T = Q^{-1}$

$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

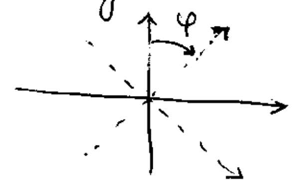
$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}^{-1} \cdot \begin{bmatrix} Z \\ W \end{bmatrix}$$

Using Theorem,

$$f_{Z,W}(z,w) = f_{X,Y}(z\cos\varphi - w\sin\varphi,\ z\sin\varphi + w\cos\varphi)$$

where $|J| = 1$.

Thus, if two RVs are rotated by an angle φ, their pdf is rotated by the same angle in the opposite direction.



ex. (Polar Representation)

Let $R = \sqrt{X^2+Y^2}$ and $\Phi = \tan^{-1} Y/X$

$r \geq 0$, $-\pi < \varphi \leq \pi$

Consider the system of equations:

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \varphi = \tan^{-1} y/x \end{cases} \Rightarrow \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \quad r > 0$$

The Jacobian is

$$J(x,y) = \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix}^{-1} = \frac{1}{r}$$

Then

$$f_{R,\Phi}(r,\varphi) = r\, f_{X,Y}(r\cos\varphi,\ r\sin\varphi),\ r>0$$
$$0,\ \text{if } r<0.$$

Auxiliary Variables:

Finding a density of mapped RVs $Z = g(X,Y)$ sometime can be simplified (by introducing) an auxiliary variable W (conveniently selected). The density of Z is then obtained by integrating W out.

ex. $Z = aX + bY$

The auxiliary RV could be $W = Y$.

11/10/2008

Functions of Multiple RVs

Similar to the case of single RV, we can define partial characterizations of a vector R.V.

1. Consider two RVs X and Y with joint pdf $f_{XY}(x,y)$. Let $Z = g(X,Y)$ ← known, deterministic

How to find $E[Z]$?

There are two options: (1) find $f_Z(z)$ and then find $E[Z]$; (2)

$$\underset{f_Z}{E}[Z] = \underset{f_{X,Y}}{E}[g(X,Y)] = \iint_{-\infty}^{+\infty} g(x,y)\, f_{X,Y}(x,y)\,dx\,dy$$

ex. Let X and Y be exponential (λ) and $X \perp\!\!\!\perp Y$. Let $M = \frac{X+Y}{2}$.

Find: $E[M]$?

$$E[X] = \frac{1}{\lambda}$$

$$\text{Then } E[M] = E\left[\frac{X+Y}{2}\right]$$

$$\overset{\text{linearity}}{=} \frac{1}{2}E[X] + \frac{1}{2}E[Y]$$

$$= \frac{1}{\lambda}$$

Let $V = \frac{1}{2}(X-M)^2 + \frac{1}{2}(Y-M)^2$

Find: $E[V]$?

$$V = \frac{(X-Y)^2}{4} \Rightarrow E[V] = \frac{1}{4}E[X^2] + \frac{1}{4}E[Y^2] - \frac{1}{2}\underbrace{E[XY]}_{\text{Requires new definition}}$$

Recall: $\text{var}(X) = \frac{1}{\lambda^2}$

Then $E[X^2] = \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = \frac{2}{\lambda^2}$

From here:

$$E[V] = \frac{1}{\lambda^2} - \frac{1}{2} E[XY]$$

Note: M and V are known as sample mean and sample variance.

2. <u>Covariance</u> is the 2nd order statistic.

Analogy can be drawn with variance of a RV. It can be used as an average measure of information that one central RV contains about some other central RV.

<u>Defn.</u> Covariance, cov(X,Y) of two RVs X and Y is defined as

$$\text{cov}(X,Y) \triangleq E[(X-E[X])(Y-E[Y])]$$

expand

$$= E[XY - \underbrace{E[X]}_{\text{constants}}Y - X\,E[Y] + E[X]E[Y]]$$

linearity

$$= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y]$$

$$= E[XY] - E[X]E[Y]$$

Thus,

$$\text{cov}(X,Y) \triangleq E[XY] - E[X]E[Y].$$

If Y is a fuct. of X, then var(X) will implicitly appear in the expression for cov(X,Y).

3. <u>Correlation Coefficient</u> (normalized covariance)

$$\rho_{X,Y} = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}}$$

$|\rho_{X,Y}| \le 1$ and thus $|\operatorname{cov}(X,Y)| \le \sqrt{\operatorname{var}(X)\operatorname{var}(Y)}$

<u>Defn.</u> Two RVs X and Y are <u>uncorrelated</u> if

$\operatorname{cov}(X,Y) = 0$,

$\rho_{X,Y} = 0$, or

$E[XY] = E[X]E[Y]$

Back to the example:

$E[XY] \underset{\text{independent}}{=} E[X]E[Y]$

$= \frac{1}{\lambda^2}$

$\Rightarrow E[V] = \frac{1}{2} \cdot \frac{1}{\lambda^2}$

<u>Defn.</u> Two RVs are <u>orthogonal</u> if

$E[XY] = 0$

$(X \perp Y)$

<u>Theorem</u>: If the two RVs, X and Y, are independent, then X and Y are uncorrelated, that is,

$f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$\Rightarrow E[XY] = E[X]E[Y]$

<u>The converse is not true</u>,

$\not\Leftarrow$.

<u>ex.</u>

$$\begin{tikzcd} & W \arrow[d] & \\ \cos X \arrow[r] & \oplus \arrow[r] & Y \end{tikzcd}$$

assume X and W have $f_{X,W}(x,w)$

Find: $E[Y]$, $\operatorname{cov}(X,Y)$

$Y = \cos X + W$

$E[Y] = E[\cos X] + E[W]$

linear operator

$\operatorname{cov}(X,Y) = E[XY] - E[X]E[Y]$

$= E[X \cdot (\cos X + W)] - E[X] \cdot (E[\cos X] + E[W])$

$= E[X \cos X] + E[X \cdot W] - E[X] \cdot E[\cos X] - E[X]E[W]$

$= E[X \cos X] - E[X] \cdot E[\cos X]$

$X \perp\!\!\!\perp W$ (uncorrelated is sufficient)

4. Joint Moments:

The joint (k,l)th moments of two RVs X and Y is defined as:

$$m_{k,l} \triangleq E[X^k Y^l] = \iint_{-\infty}^{+\infty} x^k y^l f_{X,Y}(x,y)\,dx\,dy$$

also called joint moment of order $(k+l)$.

The (k,l)th joint central moment of RVs X and Y is defined as:

$$\mu_{k,l} \triangleq E\left[(X-E[X])^k (Y-E[Y])^l\right]$$

$$= \iint (x-E[X])^k (y-E[Y])^l f_{X,Y}(x,y)\,dx\,dy$$

Clearly,

$$\mu_{10} = \mu_{01} = 0$$

$$\mu_{11} = \operatorname{cov}(X,Y), \quad \mu_{20} = \operatorname{var}(X)$$

$$\mu_{02} = \operatorname{var}(Y), \ \ldots$$

Comment: $\left.\begin{array}{l} f_{X,Y}(x,y) \\ F_{X,Y}(x,y) \end{array}\right\}$ complete characterizations of the pair (X,Y).

However, in many applications only the 1st and 2nd order moments are used.

(ex. bounds)

5. Joint Characteristic Function

Let X and Y be two RVs with the joint pdf $f_{X,Y}(x,y)$. The joint characteristic function of X and Y is defined as

$$M_{X,Y}(ju, jv) \triangleq E\left[e^{j(uX+vY)}\right]$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{jux+jvy} f_{X,Y}(x,y)\, dx\, dy$$

The inverse equation is given by:

$$f_{X,Y}(x,y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} M_{X,Y}(ju,jv) \cdot e^{-jux-jvx}\, du\, dv$$

(a) The marginal characteristic functions are:

$$M_X(ju) = M_{X,Y}(ju,jv)\Big|_{v=0}$$

$$M_Y(jv) = M_{X,Y}(ju,jv)\Big|_{u=0}$$

(b). Independence and Convolution:

If $X \perp\!\!\!\perp Y$, then $Z = X+Y$ has the following pdf:

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(\alpha) \cdot f_Y(z-\alpha)\, d\alpha$$

Then the characteristic function of Z is:

$$E[e^{jvZ}] = E[e^{jv(X+Y)}]$$

$$\underset{\substack{\uparrow \\ \text{independence, } X \perp\!\!\!\perp Y}}{=} E[e^{jvX}] \cdot E[e^{jvY}] = M_X(jv) \cdot M_Y(jv)$$

(c) Moments:

To obtain the joint moment of the order $(k+l)$:

$$m_{k,l} = \frac{\partial^k}{\partial(ju)^k} \frac{\partial^l}{\partial(jv)^l} M_{X,Y}(ju,jv)\Big|_{\substack{u=0\\v=0}} = E[X^k Y^l]$$