Continuous Models and RVs

09/18/2008

When Outcomes of random experiment take values on continuum, the RVs that map outcomes on $\mathbb{R}^1$ are of continuous type.

ex. measurements of voltage,
file download time on the Internet,
velocity and position of an aircraft on rada, etc.

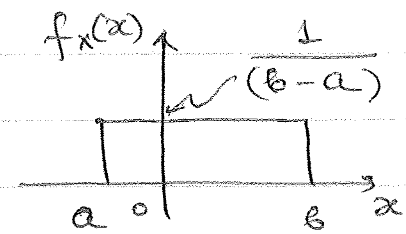
Note: The measurements are values on the real line. No need of mapping.

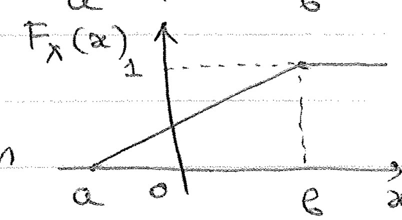
A few families of continuous RVs can be used to model a variety of cases by varying parameters of distributions.

Note: Some continuous models can be obtained as a limit of discrete models.

(1) Uniform $\sim U([a,b])$

always finite length interval





Applications:
Outcomes are equally likely.
- Homogeneous stick
- Random phase in received signal
- Recognition system with unknown object orientation

$$\Theta \sim U([0, 2\pi])$$

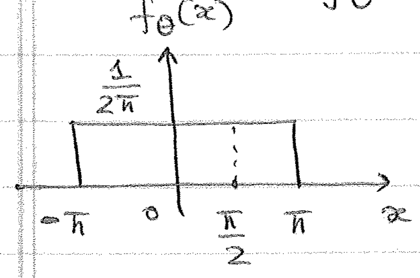
and elevation

$$\varphi \sim U([0, \tfrac{\pi}{2}]).$$

ex. In coherent communications, the phase difference between the transmitted and received signals, denoted by Θ, is modeled as $U([-\pi, \pi])$.

Find: $P[\Theta \le 0]$, $P[\Theta \le \frac{\pi}{2}]$,

$f_\Theta(x \mid \Theta \le 0)$



$$P[\Theta \le 0] = \int_{-\pi}^{0} \frac{1}{2\pi}\, dx = \frac{1}{2}$$

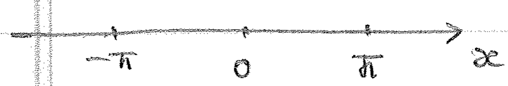
$$P[\Theta \le \frac{\pi}{2}] = \int_{-\pi}^{\pi/2} \frac{1}{2\pi}\, dx = \frac{3}{4}$$

$f_\Theta(x \mid \Theta \le 0) = ?$

We have defined the conditional cdf

$$P[\Theta \le x \mid \underbrace{\Theta \in A}_{\Theta \le 0}] \triangleq \frac{P[\Theta \le x \cap \Theta \in A]}{P[\Theta \in A]}$$

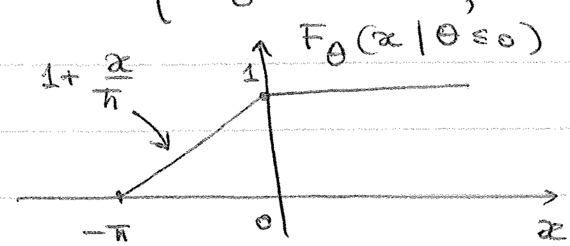
$$= \frac{P[\Theta \le x \cap \Theta \le 0]}{P[\Theta \le 0]}$$



$$= \begin{cases} P[\Theta \le x] / P[\Theta \le 0], & -\pi < x \le 0 \\ 1, & x > 0 \\ 0, & x < -\pi \end{cases}$$

Note: By conditioning on $\{\Theta \le 0\}$, we renormalize cdf and pdf

$$= \begin{cases} \int_{-\pi}^{x} \frac{1}{2\pi} d\alpha \Big/ \left(\frac{1}{2}\right), & -\pi \le x \le 0 \\ 1, & x > 0 \\ 0, & x < -\pi \end{cases}$$



The conditional pdf is

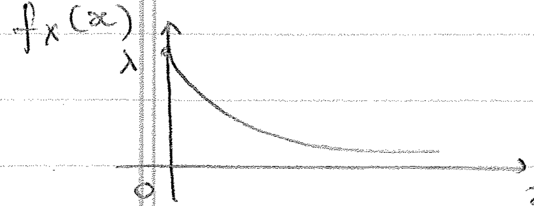
$$f_\Theta(x \mid \Theta \le 0) \triangleq \frac{d}{dx} F_\Theta(x \mid \Theta \le 0) = \frac{1}{\pi}, \quad -\pi \le x \le 0,$$

$$0, \quad \text{elsewhere.}$$

(2) Exponential pdf

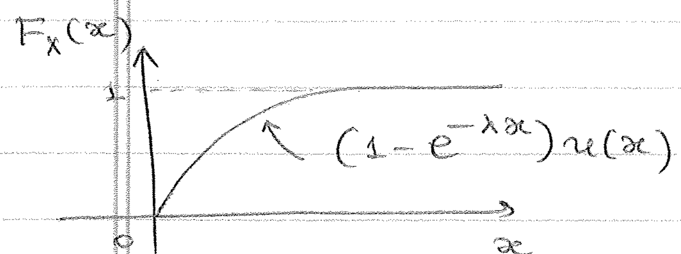
The exponential RV with parameter λ is

$$f_X(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & x \geq 0 \end{cases}$$



$$F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\, d\alpha$$

$$= \begin{cases} 0, & x < 0 \\ \int_0^x \lambda e^{-\lambda \alpha} d\alpha = 1 - e^{-\lambda x} \end{cases}$$



Often arises as a fnct. of other RVs.

ex. $U \sim$ uniform $(0,1)$, then $X = \ln(1/U)$ is exponential with $\lambda = 1$.

Note: λ is the rate of event

$\lambda_2 > \lambda_1$

more concentrated density

Applications:

- Modeling time between occurrence of two events

ex. time between two decays of radioactive substance.

- Modeling of lifetime of a device or system (reliability)
- Length of all phone calls.

Comment: Exponential RV can be obtained as a limit of the geometric RV.

See example on P. 30

(3.) Gaussian R.V., $X \sim N(m, \sigma^2)$

parameters

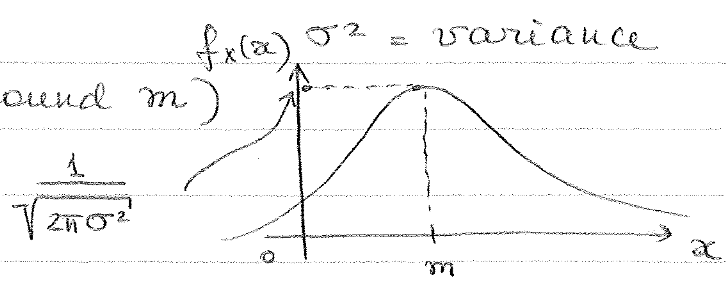
m = mean

σ^2 = variance

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

(symmetric around m)

ex. target



30/ ex. (exponential RV)

Suppose that the probability that the cell phone call ends by $t+\Delta t$, given that it lasted more than t seconds, is approximately $\lambda \Delta t$ (Δt is small).

Show that the call duration is exponential RV with parameter λ.

Proof: Let T be the call duration.

Then

$$\underbrace{P[T \le t+\Delta t \mid T>t]}_{\frac{P[T\le t+\Delta t \,\cap\, T>t]}{\underbrace{P[T>t]}_{\frac{P[t<T\le t+\Delta t]}{P[T>t]}}}} \approx \lambda \cdot \Delta t \qquad (*)$$

$$\frac{P[t<T\le t+\Delta t]}{P[T>t]} = \frac{\int_t^{t+\Delta t} f_T(x)\,dx}{P[T>t]}$$

The left part of (*) can be approximated by

$$\frac{f_T(t)\cdot \Delta t}{P[T>t]} = \lambda \Delta t .$$

From here $$f_T(t) = \lambda \cdot \int_t^{+\infty} f_T(x)\,dx \qquad (**)$$

Differentiate both sides of (**):

$$f_T'(t) = -\lambda f_T(t), \quad t>0 .$$

$$f_T(t) = c\, e^{-\lambda t}$$

$c = \lambda$ (for $f_T(t)$ to be a density).