1.

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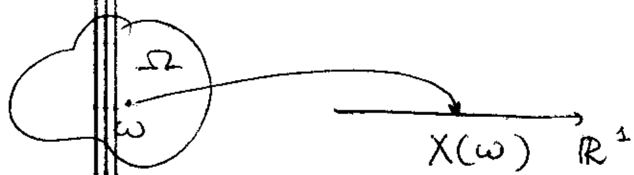
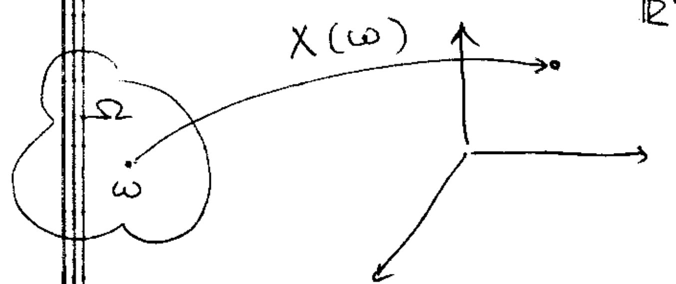
Summary: To specify a random experiment $(\Omega, \mathcal{F}, \mathcal{P})$ ~ probability space.

A RV $X(\omega)$ is a function that maps elements of Ω onto the real line



A random vector $\underline{X}(\omega)$ is a mapping of elements of Ω onto $\mathbb{R}^n$



Events:

$A = \{\omega : X(\omega) \leq x\} \in \mathcal{F}$ for all $x \in \mathbb{R}^1$

Complete Characterization:

$$P_X(x) = \Pr[\omega : X(\omega) \leq x]$$
for all x

Vectors: Product type events.

Partial Characterizations:

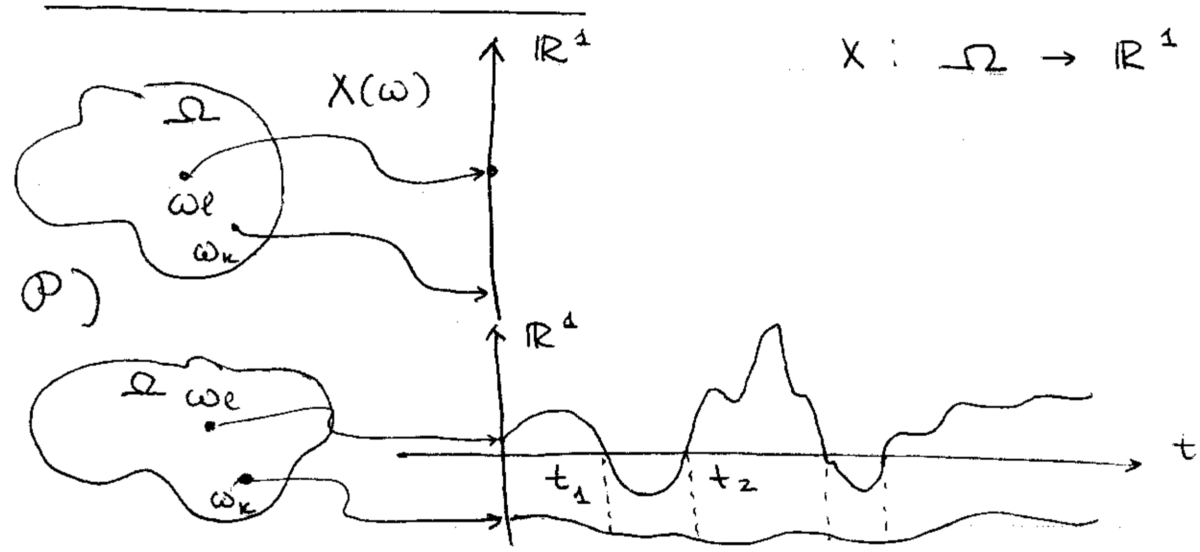
m_x, σ_x^2, moments
$E[X^k]$, $k = 0, 1, 2, \ldots$

Random Processes

R.V.

$X : \Omega \to \mathbb{R}^1$

$(\Omega, \mathcal{F}, \mathcal{P})$



$$X : \underbrace{\Omega \times T}_{\text{domain}} \rightarrow \underset{\substack{\text{range}\\ \text{space}}}{S} \qquad \begin{matrix} \mathbb{R}^1 \\ \mathbb{R}^2 \\ \vdots \end{matrix}$$

ex (index set)

discrete-time process

$$T = \{t_1, t_2, t_3, \ldots, t_n, \ldots\} \quad \text{countable}$$

$$T = \{t,\ t \in \mathbb{R}^1\} \quad \text{random signal}$$

A Random Process $X(t)$ is specified iff a rule is given or implied for determining the joint c.d.f. $P_{\underline{X}(t)}$ for any finite set of observation instances $(t_1, \ldots, t_n)$.

$$T = \{t : t \in \mathbb{R}^2\} \quad \text{2D-space}$$

$$T = \{t : t \in \mathbb{R}^2 \times \mathbb{R}^1\} \quad \text{picture change}$$

Terminology: $X(t, \omega)$

for t fixed	random variable
for ω fixed	sample function
t and ω vary	ensemble

Sample process at $t_1, t_2, \ldots, t_n$, then

$$\underbrace{X(t_1, \omega),\ X(t_2, \omega),\ \ldots,\ X(t_n, \omega)}_{\text{joint RVs}}$$

Characterization: $X(t, \omega)$ is completely characterized if the joint c.d.f.

$$\Pr[X(t_1) \le x_1,\ X(t_2) \le x_2,\ \ldots,\ X(t_n) \le x_n]$$
$$= P_{X(t_1), X(t_2), \ldots, X(t_n)}(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n)$$

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is known for all finite joint sets $(t_1, t_2, \ldots, t_n) \in T$ and all $x_1, x_2, \ldots, x_n \in S$.

Specifications of R.P.s

(1) Rule of construction for the cdf

$$P_{\underline{X}(t)}(\underline{x})$$

(2) Parameterization

$$X(\omega, t) = \cos(2\pi f t + \Theta(\omega))$$

difficult, but possible to find $P_{\underline{X}(t)}(\underline{x})$ via $f_\Theta(\theta)$.

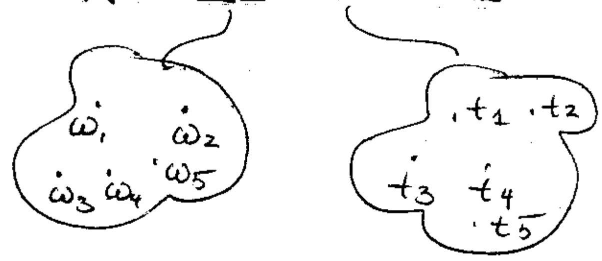
RV $\sim f_\Theta(\theta)$

(3) transformation of a completely characterized R.P.

usually possible for G.R.P.

Random Process:

$$X : \Omega \times T \to \mathbb{R}^n$$



$X(\omega, t)$

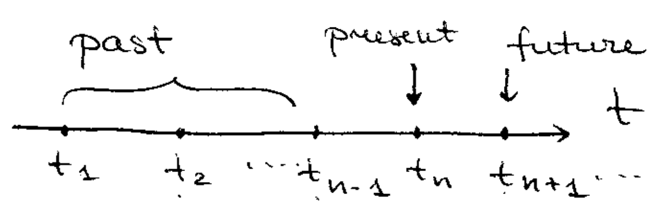
ex. ("rule") Markov Process

$$X : \Omega \times T \to S$$

S ← state space

T ← parameter space (index set)

T (param. space) \ S	discrete	continuous
discrete	discrete Markov chain	discrete parameter Markov process
continuous	Continuous parameter Markov chain	Continuous parameter Markov process



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(4).

Defn. $X(t)$ is a Markov process if

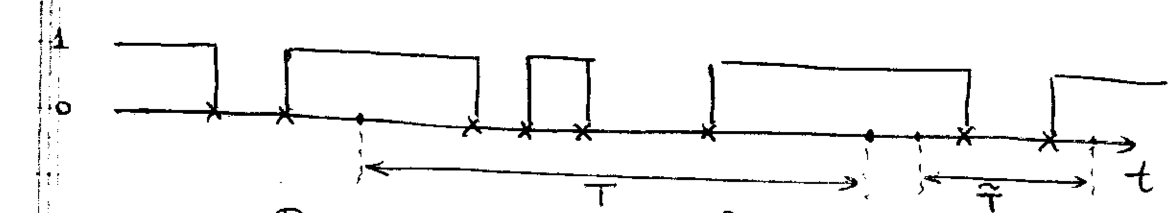
$$\Pr[X(t_n) \le x_n \mid X(t_{n-1}) \le x_{n-1}, \ldots, X(t_1) \le x_1]$$
$$= \Pr[X(t_n) \le x_n \mid X(t_{n-1}) \le x_{n-1}].$$

Then the joint distribution:

$$\Pr[X(t_1) \le x_1, X(t_2) \le x_2, \ldots, X(t_n) \le x_n]$$
$$= \Pr[X(t_n) \le x_n \mid X(t_{n-1}) \le x_{n-1}]$$
$$\times \Pr[X(t_{n-1}) \le x_{n-1} \mid X(t_{n-2}) \le x_{n-2}]$$
$$\times \ldots \times \Pr[X(t_2) \le x_2 \mid X(t_1) \le x_1] \cdot \underbrace{\Pr[X(t_1) \le x_1]}_{\text{marginal cdf}}.$$

Specify:
- Type of Markov process
- Transition probabilities
- Marginal cdf.

ex. 2: ("rule") Random Telegraph Wave



$X(t)$ takes '0' or '1'

Rules:

1. $\Pr[X(t) = 0] = \Pr[X(t) = 1] = \frac{1}{2}$
2. Changes $0 \to 1$ or $1 \to 0$ occur at each event.
3. $\Pr[k$ events in the interval $T]$ $= \frac{(\lambda T)^k}{k!} \cdot e^{-\lambda T}$

($\lambda \sim$ average intensity of events)

4. the number of events in disjoint intervals are statistically independent.

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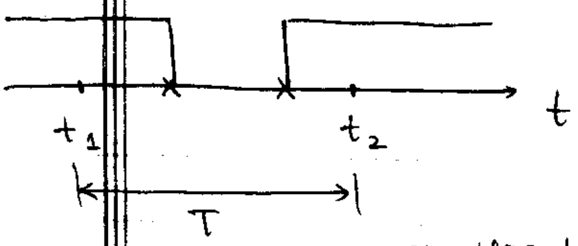
(5).



Find: $Pr[X(t_2) = 1 \mid X(t_1) = 1;\ t_1, t_2]$

$$= Pr[\text{even \# events in } [t_1, t_2]]$$

$$= \sum_{k \text{ even}} \frac{(\lambda T)^k}{k!} \cdot e^{-\lambda T}$$

use this trick

$$= \sum_{k=0}^{+\infty} \frac{1}{2}[1 + (-1)^k] \frac{(\lambda T)^k}{k!} \cdot e^{-\lambda T}$$

$$= \frac{e^{-\lambda(t_2 - t_1)}}{2} \cdot \left(\underbrace{\sum_{k=0}^{+\infty} \frac{(\lambda T)^k}{k!}}_{e^{\lambda T}} + \underbrace{\sum_{k=0}^{+\infty} \frac{(-\lambda T)^k}{k!}}_{e^{-\lambda T}} \right)$$

$$= \frac{1}{2}[1 + e^{-2\lambda(t_2 - t_1)}]$$

We can also find: $Pr[X(t_2) = 0 \mid X(t_1) = 1]$

$$= \frac{1}{2}[1 - e^{-2\lambda(t_2 - t_1)}]$$

$Pr[X(t_2) = 1 \mid X(t_1) = 0]$

$Pr[X(t_2) = 0 \mid X(t_1) = 0]$

Consider: $Pr[X(t_3) = 1, X(t_2) = 1, X(t_1) = 1]$

$$= Pr[X(t_3) = 1 \mid X(t_2) = 1, X(t_1) = 1]$$

$$\times Pr[X(t_2) = 1 \mid X(t_1) = 1] \cdot \underbrace{Pr[X(t_1) = 1]}_{1/2}$$

$Pr[X(t_3) = 1 \mid X(t_2) = 1]$

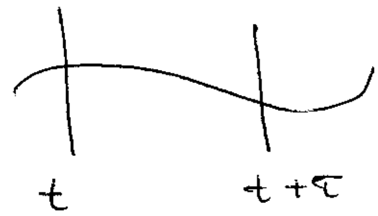
$$= \left(\frac{1}{2}\right)^3 \cdot [1 + e^{-2\lambda(t_3 - t_2)}][1 + e^{-2\lambda(t_2 - t_1)}]$$

Defn.: A RP $X(t)$ is a strict sense stationary of order 1 if

$$P_{X(t+\tau)}(x; t+\tau) = P_{X(t)}(x; t)$$

⑥.

A sample function



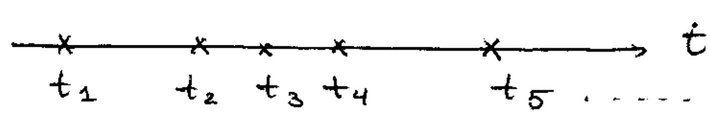
A RP is <u>SSS of order 2</u> if

$$P_{X(t_1+\tau), X(t_2+\tau)}(x_1, x_2 ; t_1+\tau, t_2+\tau)$$
$$= P_{X(t_1), X(t_2)}(x_1, x_2 ; t_1, t_2)$$

⋮

A <u>RP is SSS</u> if it is SSS for all orders.

<u>ex.</u> Random Telegraph Wave



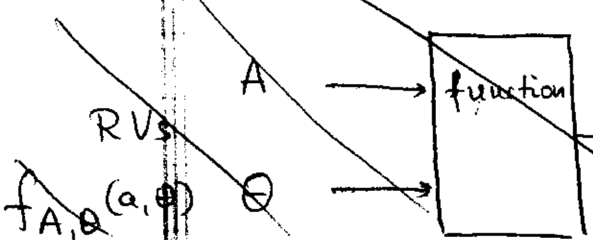
$$\Pr[X(t_n)=1, X(t_{n-1})=1, \ldots, X(t_1)=1]$$
$$= \frac{1}{2^n} \cdot \prod_{i=2}^{n} \left[1 + e^{-2\lambda(t_i - t_{i-1})}\right]$$

Consider a shift by τ :

$$\frac{1}{2^n} \prod_{i=2}^{n} \left[1 + e^{-2\lambda(t_i + \tau - t_{i-1} - \tau)}\right]$$

$\Rightarrow$ The RP is SSS.

<u>ex.</u> ~~(parameterization)~~ <u>Specifying a RP via parameterization</u>



~~$X(t, \omega) = A(\omega) \times \cos(2\pi f_0 t + \Theta(\omega))$~~ ← RVs

$f_{A,\Theta}(a, \theta)$ ~~need~~ joint pdf

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