Play important role in designing linear stochastic filters and predictors

⑦ Partial Characterization of a R.P. $X(t,\omega)$

- DC component
- Whether correlated
- Linear Systems with stochastic inputs.

1. Mean Value Function

$$m_X(t) = E[X(t)] \triangleq \int_{-\infty}^{+\infty} x \cdot f_{X(t)}(x)\,dx$$

ex. 1:

$$X(t) = A\cos(2\pi f_0 t + \theta)$$

Function of 2 RVs

$f_{A,\theta}(a,\theta)$

$$m_X(t) = \int_{-\infty}^{+\infty}\int_{0}^{2\pi} a\cos(2\pi f_0 t + \theta)\cdot f_{A,\theta}(a,\theta)\,da\,d\theta$$

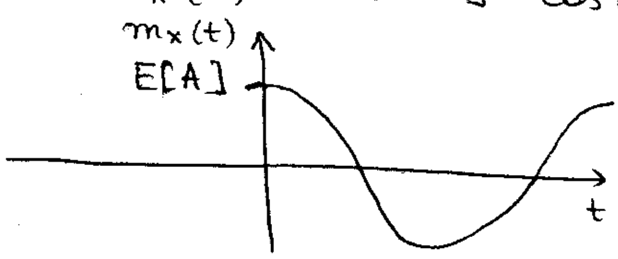
general

$$\overset{A \perp\!\!\!\perp \Theta}{=} E[A]\cdot \int_0^{2\pi} \cos(2\pi f_0 t + \theta)\, f_\Theta(\theta)\,d\theta$$

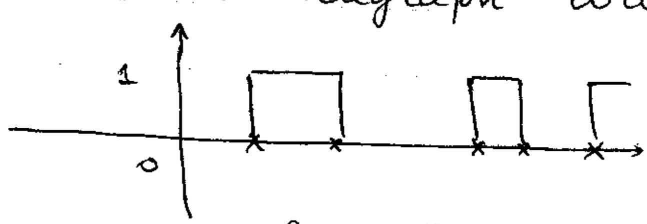
$$\overset{A \perp\!\!\!\perp \Theta,\ \Theta \sim U[0,2\pi]}{=} 0$$

If $\theta = 0$ (a constant) $f_\theta(\theta) = \delta(\theta)$, then

$$m_X(t) = E[A]\cdot\cos(2\pi f_0 t)$$



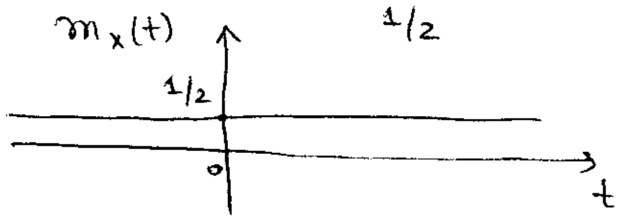
ex. 2: Random Telegraph Wave:



← a sample path

$$m_X(t) = E[X(t)] = \underbrace{1\cdot \Pr[X(t)=1]}_{1/2} + 0\cdot\Pr(X(t)=0]$$

$$= \frac{1}{2}$$



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8.

Correlation Functions:

Let $X(t)$ be a real valued process.

$X : \Omega \times T \to S$, the autocorrelation function:

sample space, index space, range of the process

$$R_X(t,u) \triangleq E[X(t) \cdot X(u)]$$

(second moment)

function of t and u in general

For $X(t) \sim$ complex valued

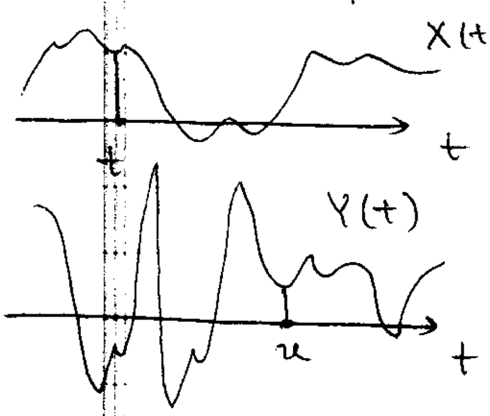
$$R_X(t,u) \triangleq E[X(t) \cdot X^*(u)]$$

$Re\{X(u)\} - j\,Im\{X(u)\}$

complex conjugate

$Re\{X(t)\} + j\,Im\{X(t)\}$

The cross-correlation function is defined for two processes $X(t)$ and $Y(t)$ as



$$R_{X,Y}(t,u) = E[X(t) \cdot Y(u)]$$

real valued processes

$$= E[X(t)\, Y^*(u)]$$

complex valued processes

In general we need to know joint probability

ex. $X(t) = A \cos(2\pi f_0 t + \theta)$ real valued process

$A \perp\!\!\!\perp \theta$, $\theta \sim U[0, 2\pi]$

The autocorrelation function:

$$R_X(t,u) = E[X(t)\, X(u)]$$

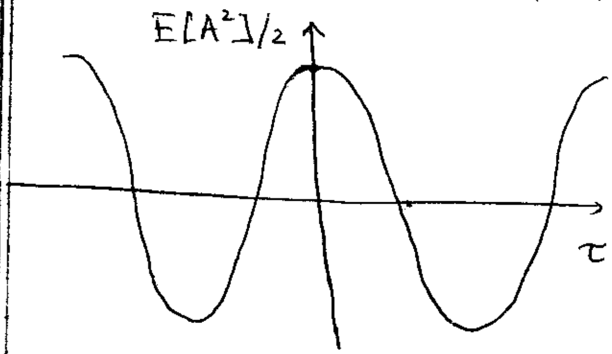
$$= E[A^2 \cos(2\pi f_0 t + \theta) \cos(2\pi f_0 u + \theta)]$$

[redacted]

$$= E[A^2] \cdot E\left[\frac{1}{2} \cos(2\pi f_0 (t+u) + 2\theta) + \frac{1}{2} \cos(2\pi f_0 (t-u)) \right]$$

$$= \frac{E[A^2]}{2} \cdot \cos(2\pi f_0 \underbrace{(t-u)}_{\tau})$$

Note that $R(t,u)$ is a function of $t-u=\tau$



Note:

(1) $R_X(\tau)$ depends only on τ

(2) $R_X(\tau)$ is even

(3) $R_X(\tau)$ is continuous in τ

(4) $|R_X(\tau)| \le R_X(0)$

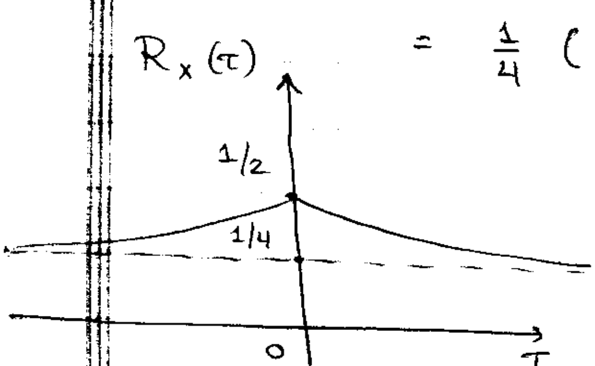
ex. Random Telegraph Wave

$$R_X(t,u) = E[X(t) \cdot X(u)]$$

$$= 1 \cdot \Pr[X(t)=1, X(u)=1] + 0 \times \Pr \ldots\ldots$$

$$= 1 \cdot \Pr[X(u)=1 \mid X(t)=1] \cdot \Pr[X(t)=1]$$

$$= \frac{1}{4}\left(1 + e^{-2\lambda \underbrace{|t-u|}_{\tau}}\right)$$



(1) depends on τ

(2) even function of τ

(3) continuous

(4) $|R_X(\tau)| \le R_X(0)$

3 <u>Covariance Function</u> ($X(t) \sim$ real valued)

$$K_X(t,u) \triangleq E[(X(t) - m_X(t))(X(u) - m_X(u))]$$

[redacted]

1/2

(Insertion prior to LTI systems)

Linear Processing of Random Signals

In a variety of practical applications, measurements (observed data), can be modeled as samples of finite realizations of stochastic processes.

For simplicity of analysis and design many processes are treated as stationary or $\underbrace{\text{WSS}}$.

stationary: good 1st order approximation

WSS: known mean and correlation functions only; a good first order approximation in many cases.

WSS processes are characterized by:

$$E[X(t)] = m = \text{const}$$

$$E[X(t)\,X(u)] = R_{XX}(\underbrace{t-u}_{\tau})$$

The correlation function $R_{XX}(\tau)$ indicates the rate of change of the process $X(t)$ (how fast samples of $X(t)$ loose "linear" information about one another ($X(u)$ about $X(t)$)).

In engineering, it is common to analyze behaviour of functions or discrete samples in a transform domain. The Fourier domain is the most popular tool for analysis.

Dealing with the spectrum of $X(t)$ does not make sense. However, dealing with $\mathcal{F}\{R_{XX}(\tau)\}$

deterministic functions

provides a lot of intuition.

[redacted]

<u>Power Spectral Density</u> (PSD) is defined as the Fourier transform of $R_{XX}(\tau)$ (WSS process):

$$S_X(f) = \int_{-\infty}^{+\infty} R_{XX}(\tau) \cdot e^{-j2\pi f t} \, d\tau$$

<u>Properties</u> of $S_X(f)$: If $X(t)$ is real-valued, WSS, then $R_{XX}(\tau)$ is symmetric.

- $S_X(f)$ is real-valued
- — " — symmetric
- $S_X(f) \geq 0$ for all f
- $R_{XX}(0) = \underbrace{E[\,|X(t)|^2\,]}_{\text{power}} = \int_{-\infty}^{+\infty} S_X(f)\, df$

With these tools we can now approach the problem of linear processing of observed samples or realization of RP.

We initially focus of <u>Linear time invariant</u> (LTI) systems and filters.

<u>Defn.</u>: A process $X(t)$ is a WSS process if

1. $m_X(t) = \text{const}$
2. $R_X(t,u) = R_X(t-u,0) = R_X(\tau)$ and $K_X(t,u) = K_X(t-u,0) = K_X(\tau)$

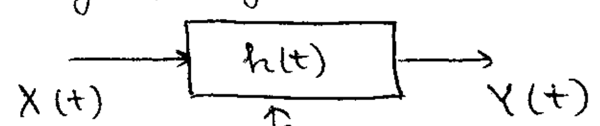
(See Insertion 1-2)

Random Processes in LTI Systems

(linear time-invariant)

Many applications involve processing of signals

- filtering
- prediction
- modulation, coding
- signal processing,
- estimation
- detection



$X(t)$ WSS with finite power $E[|X(t)|^2] < \infty$ ← bounded input

impulse response

$$Y(t) = \int_{-\infty}^{+\infty} X(\tau) \times h(t-\tau)\, d\tau$$

Note: need to ensure that the designed system is <u>stable</u>, in BIBO sense.

BIBO stability

What is the condition for this?

$$\left| E[|Y(t)|^2] \right| = \left| \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(t-\tau_1)\, \underbrace{E[X(\tau_1)X^*(\tau_2)]}_{R_X(\tau_1-\tau_2)} \times h(t-\tau_2)\, d\tau_1\, d\tau_2 \right|$$

$$\leq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |h(t-\tau_1)|\,|R_X(\tau_1-\tau_2)|\,|h(t-\tau_2)|\, d\tau_1\, d\tau_2$$

$$\leq R_X(0) \cdot \underbrace{\left[\int_{-\infty}^{+\infty} |h(t-\tau_1)|\, d\tau_1 \right]^2}_{<\infty}$$

by properties of R_X

[redacted]

The necessary and sufficient conditions for BIBO is

$$\int_{-\infty}^{+\infty} |h(t)|\,dt < \infty$$

Let us show if $\frac{X(\cdot) \text{ is WSS}}{Y(\cdot) \text{ is WSS}}$, then

$$\begin{cases} m_Y = m_X\, H(0) \\ R_Y(\tau) = \displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\alpha)\, R_X(\tau - \alpha + \beta)\, h(\beta)\, d\alpha\, d\beta \end{cases}$$

<u>Frequency Domain Method</u> :

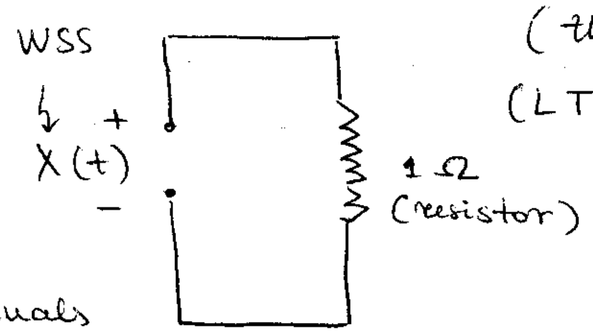
Let $X(\cdot) \sim$ WSS with the correlation function $R_X(\tau)$.

Define the power spectrum of $X(\cdot)$ as

$$S_X(f) = \int_{-\infty}^{+\infty} R_X(\tau) \cdot e^{-j2\pi f \tau}\, d\tau$$

ex

voltage $X(t)$ is applied across the terminals



(the meaning of $S_x(f)$)

(LTI system)

Instant Power

$|X(t)|^2$

Average Power

$$P_{avg.\,(2T)}(\omega) = \frac{1}{2T}\int_{-T}^{T} |X(t)|^2\, dt$$

interval

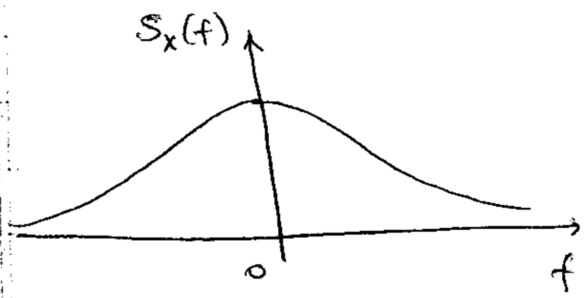
Expected Power :
(in watts)

[redacted]

(12)

$$\bar{P}(t) = E\left[\,|X(t)|^2\right] = R_X(0) = \int_{-\infty}^{+\infty} S_X(f)\,df$$

$S_X(f)$: power spectrum density (watts / Hz)



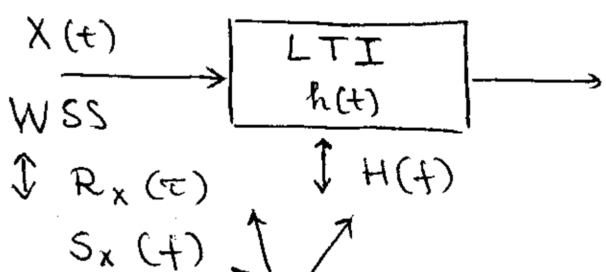
a sketch of a typical psd

Suppose $X(t) \sim$ WSS, real valued process

Properties of psd:

- $S_X(f)$ is even
- nonnegative
- real

Consider an LTI system:



$H(0) < \infty$ bounded system

Correlation function of the output:

$$R_Y(\tau) = E\left\{ \int_{-\infty}^{+\infty} X(\alpha)\cdot h(t-\alpha)\,d\alpha \times \int_{-\infty}^{+\infty} X^*(\beta)\cdot h^*(t+\tau-\beta)\,d\beta \right\}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underbrace{h(t-\alpha)}_{u}\; \underbrace{E\left[X(\alpha)\cdot X^*(\beta)\right]}_{R_X(\tau - v + u)} \cdot h^*(\underbrace{t+\tau-\beta}_{v})\,d\alpha\,d\beta$$

$\alpha = t-u$, $\beta = t+\tau-v$

(13).

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(u)\left[\int_{-\infty}^{+\infty} S_x(f)\cdot e^{j2\pi f\cdot(\tau-v+u)}\,df\right] h^*(v)\,du\,dv$$

exchange integrals

$$= \int_{-\infty}^{+\infty} S_x(f)\cdot\left[\int_{-\infty}^{+\infty} h(u)\cdot e^{j2\pi f u}\,du\right]\left[\int_{-\infty}^{+\infty} h^*(v)\,e^{-j2\pi f v}\,dv\right] \times e^{j2\pi f\tau}\,df$$

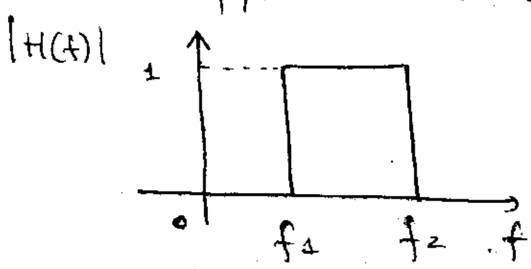
$$= \int_{-\infty}^{+\infty} S_x(f)\cdot \underbrace{H(f)\cdot H^*(f)}_{|H(f)|^2}\cdot e^{j2\pi f\tau}\,df$$

$$= \int_{-\infty}^{+\infty} S_Y(f)\cdot e^{j2\pi f\tau}\,df$$

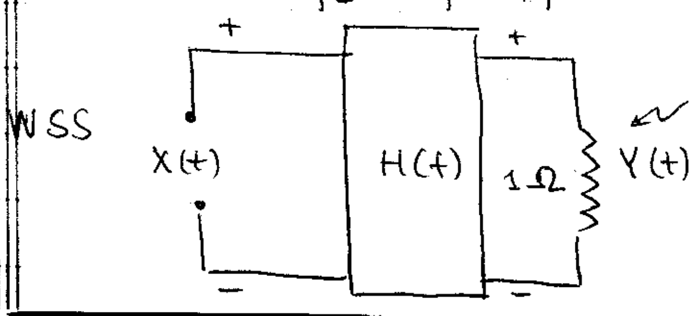
$$\Rightarrow \quad R_Y(\tau) = \int_{-\infty}^{+\infty} \underbrace{|H(f)|^2\cdot S_x(f)}_{S_Y(f)}\,e^{j2\pi f\tau}\,df$$

$$S_Y(f) = |H(f)|^2\cdot S_x(f)$$

ex. Suppose that the frequency response is



~ ideal band pass filter



Average power dissipation on 1 Ω resistor

(14).

$$\overline{P}_{Avg.} = E\left[\,|Y(t)|^2\,\right] = \int_{-\infty}^{+\infty} S_Y(f)\,df$$

$$= \int_{-\infty}^{+\infty} |H(f)|^2 \cdot S_X(f)\,df = \int_{f_1}^{f_2} S_X(f)\,df$$