Medical Image Analysis

CS 593 / 791

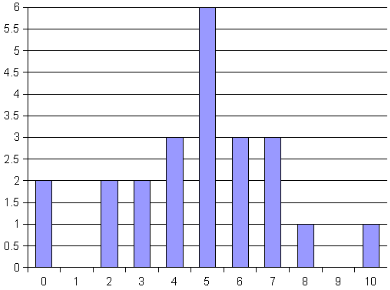
Computer Science and Electrical Engineering Dept.
West Virginia University

10th January 2007

Outline

Outline

Placement Quiz Results



Histogram of Scores

Using sliding windows for image processing

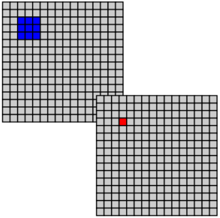


Figure: Pixel [i,j] in the output image (shown in red) depends on the pixels in the neighborhood of [i,j] in the input image (shown in blue).

(i-1, j+1)	(i, j+1)	(i+1, j+1)
(i-1, j)	(i, j)	(i+1, j)
(i-1, j-1)	(i, j-1)	(i+1, j-1)

Coordinates in the 3×3 neighborhood of pixel (i,j).

Image smoothing

We can compute several things within each window:

- Local averaging : mean
- Weighted local average : weight the center pixel higher
- Nonlinear filtering : median

- We can look at larger neighborhoods : 5×5, 7×7 and larger.
- Why are these odd dimensions?
- In 3D : $3 \times 3 \times 3$ neighborhoods and larger.

Local averaging

We associate weights, w_{km} with each voxel in the sliding window:

$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

The new intensity, g(i,j), computed from from the original image, f, and the weights, w

$$g(i,j) = \sum_{m=-1}^{1} \sum_{k=-1}^{1} w_{km} f(i+k, j+m)$$

is the mean of intensity values within the sliding window.

When applied to the whole image we write $g = w \otimes f$

Local averaging results

- Edges are destroyed.
- Smoothing is a low-pass filtering operation.
- High frequency components are attenuated, low frequency components of the image are preserved.

It can be shown that if image f is corrupted by additive Gaussian noise of variance σ_f^2 then the variance of the noise on image g is

$$\sigma_g^2 = \sum_{m=-1}^{1} \sum_{k=-1}^{1} w_{km}^2 \sigma_f^2$$

Fourier Transform - Convolution Theorem

The continuous definition of convolution

$$f * g = \int_{-\infty}^{\infty} f(\tau) g(x - \tau)\, d\tau \tag{1}$$

$$\mathcal{F}[f * g] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) g(x - \tau)\, d\tau \right] e^{-2\pi i k x}\, dx \tag{2}$$

Using the fact that $e^x = e^{\tau} e^{x-\tau}$

$$\mathcal{F}[f * g] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [e^{-2\pi i k \tau} f(\tau)][e^{-2\pi i k (x-\tau)} g(x - \tau)]\, dx\, d\tau \tag{3}$$

Substituting $y = x - \tau$, and $dy = dx$

$$\mathcal{F}[f * g] = \int_{-\infty}^{\infty} e^{-2\pi i k \tau} f(\tau)\, d\tau \int_{-\infty}^{\infty} e^{-2\pi i k y} g(y)\, dy \tag{4}$$

$$\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g] \tag{5}$$

- Convolution can be implemented as multiplication in the frequency domain.
- Gaussian convolution is Gaussian multiplication in frequency domain.

Image Gradient

$$\nabla I(x,y) = \begin{bmatrix} \frac{\partial I}{\partial x} \\ \\ \frac{\partial I}{\partial y} \end{bmatrix} \qquad (6)$$

Vector points in the direction of fastest increase in $I(x,y)$

- Edges are discontinuities in intensity
- $||\nabla I||$ is the edge "strength" (sharpness of the edge)
- ∇I is perpendicular to the edge

Edge detection using the gradient

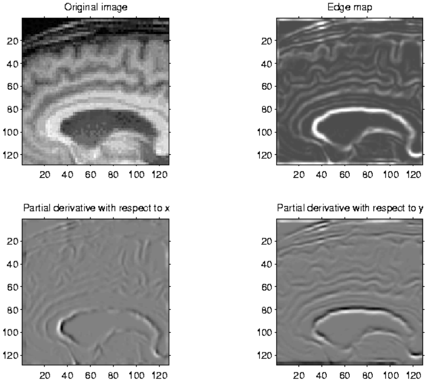


Image from ”Visualizing calculus: The use of the gradient in image processing”

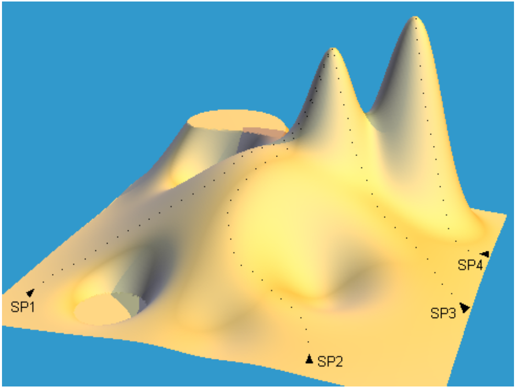
(http://amath.colorado.edu)

Gradient Descent

The gradient is useful in this simple numerical optimization technique

$$\frac{dx}{dt} = \mp \nabla I \tag{7}$$

Will converge to a local minimum/maximum of I.



Divergence

$$\text{div}\,\mathbf{V}(\mathbf{x},\mathbf{y}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \tag{8}$$

where the vector-valued function $\mathbf{V}(\mathbf{x},\mathbf{y}) = \begin{bmatrix} v_1(x,y) \\ v_2(x,y) \end{bmatrix}$

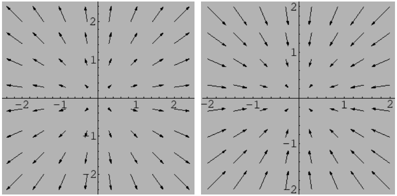
More notation:

$\text{div}(\nabla I) = \nabla \cdot \nabla I = \nabla^2 I$ (the Laplacian)

Divergence of a vector field

$$\text{div}\,\mathbf{V}(\mathbf{x},\mathbf{y}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \tag{9}$$

Physical interpretation : sources or sinks a velocity field.



div $V > 0$ (left), div $V < 0$ (right)
Image from "The idea of divergence and curl"

(http://www.math.umn.edu)

Taylor Series

$$f(x) \approx f(a) + (x-a)f'(x) + \frac{1}{2}(x-a)^2 f''(x) + \ldots \frac{1}{n!}(x-a)^n f^{(n)}(x) \quad (10)$$

Taylor series expansion is the basis for many numerical methods. For example Newton's method...

The Hessian Matrix

The Hessian is a matrix of second partial derivatives of the function, $f(x, y)$:

$$\mathbf{H}(f)(a) = \begin{bmatrix} f_{xx}(a) & f_{xy}(a) \\ f_{yx}(a) & f_{yy}(a) \end{bmatrix} \tag{11}$$

Conditions for minimizing $f(x)$:

- $f'(x) = 0$
- $f''(x) > 0$

Conditions for minimizing $f(x, y)$:

- $\nabla f(x, y) = \mathbf{0}$
- $\det(H(f)(x, y)) > 0$

Multivariable Taylor Series

$$f(x) \approx f(x_0) + (x - x_0)^T \nabla f(a) + \frac{1}{2}(x - x_0)^T \mathbf{H}(f)(a)(x - x_0) \tag{12}$$

Taylor series expansion is the basis for many numerical methods. For example Newton's method...

Geometry

Parametric Circle

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos(k\theta + \theta_0) \\ r\sin(k\theta + \theta_0) \end{bmatrix}$$
$$\theta \in [0, 2\pi]$$

Implicit circle

$$x^2 + y^2 - r^2 = 0$$

Level set methods involve embedding curves and surfaces in images:

- $I(x, y) = c$: implicit curve
- $I(x, y, z) = c$: implicit surface

Evolving the image, $I(x, y)$, results in an evolving curve/surface

Vector and function norms

- $||v||_1 = |x| + |y| + |z|$
- $||v||_2 = \sqrt{x^2 + y^2 + z^2}$
- $||v||_\infty = \max(|x|, |y|, |z|)$

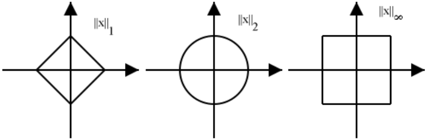


Figure: Equidistant points under various norms.

The generalized vector p-norm is $(\sum_i |x_i|^p)^{1/p}$

The p-norm of the function, $f(x)$ is given by

$$L_2(f) = \left[\int_\Omega |f(x)|^p dx\right]^{1/p} \tag{13}$$

Application to image enhancement

Energy functionals will often take the form of the norm of some function

A smooth image, I, may be found by minimizing

$$E(I) = \int_{\Omega} ||\nabla I(x)||^p dx \tag{14}$$

- What sort of images have small values of E?
- Later we will see that as $p \rightarrow 1$ edges are preserved.

The data constraint functional is given by

$$E_d(I) = \int_{\Omega} (I(x) - I_0(x))^2 dx \tag{15}$$

Least Squares

For solving overconstrained linear systems : minimize the squared error

$$\min_x ||\mathbf{Ax} - \mathbf{b}||^2 = \min_x (\mathbf{Ax} - \mathbf{b})^T (\mathbf{Ax} - \mathbf{b}) \tag{16}$$

$$= \min_x (\mathbf{x}^T \mathbf{A}^T - \mathbf{b}^T)(\mathbf{Ax} - \mathbf{b}) \tag{17}$$

$$\frac{d}{d\mathbf{x}}(\mathbf{x}^T \mathbf{A}^T \mathbf{Ax} - \mathbf{x}^T \mathbf{A}^T \mathbf{b} - \mathbf{b}^T \mathbf{Ax} + \mathbf{b}^T \mathbf{b}) = 0 \tag{18}$$

$$2\mathbf{A}^T \mathbf{Ax} - 2\mathbf{A}^T \mathbf{b} = 0 \tag{19}$$

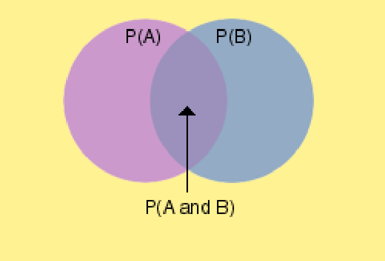
Least Squares

So,

$$\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} \tag{20}$$

solves the overconstrained system.
$(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ is sometimes called the pseudo-inverse

Bayes' Rule



$$P(A|B) = \frac{P(A,B)}{P(B)} \tag{21}$$

$$P(B|A) = \frac{P(A,B)}{P(A)} \tag{22}$$

Bayes' Rule

So, $P(A, B) = P(B)P(A|B) = P(A)P(B|A)$,
which we can rearrange as:

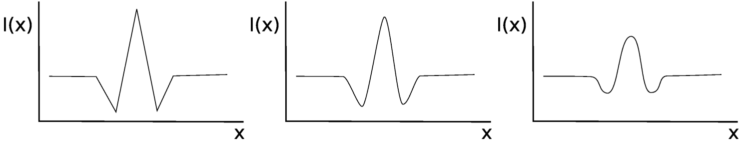
$$P(A|B) = \frac{P(A)P(B|A)}{P(B)} \tag{23}$$

Outline

The Heat Equation / Diffusion Equation

$$\frac{\partial I(x,y,z)}{\partial t} = \mathrm{div}(\nabla I(x,y,t)) \qquad (24)$$

- The equation describes the way physical systems achieve equilibrium.
- When describing heat transfer, $I(x)$ is temperature.
- When describing diffusion, $I(x)$ is molecular concentration.



Solving the heat equation

In one dimension:

$$\frac{\partial I}{\partial t} = \frac{\partial^2 I}{\partial x^2} \tag{25}$$

Take the Fourier transform of both sides

$$\mathcal{F}\left[\frac{\partial I}{\partial t}\right] = \mathcal{F}\left[\frac{\partial^2 I}{\partial x^2}\right] \tag{26}$$

Let $\mathcal{F}[I(x,t)] = U(\omega, t)$ and

Recall that

if $f(x) \leftrightarrow F(\omega)$, then $f'(x) \leftrightarrow i\omega F(\omega)$, and
$f''(x) \leftrightarrow -\omega^2 F(\omega)$

The Heat Equation / Diffusion Equation

$$\frac{\partial U(\omega,t)}{\partial t} = -\omega^2 U(\omega,t) \tag{27}$$

with initial conditions

$$U(\omega,0) = U_0(\omega) \tag{28}$$

You can verify that

$$U(\omega,t) = U_0(\omega)e^{-\omega^2 t} \tag{29}$$

Is a solution to the problem.

The Heat Equation / Diffusion Equation

Taking the inverse Fourier transform

$$\mathcal{F}^{-1}[U(\omega, t)] = \mathcal{F}^{-1}[U_0(\omega)e^{-\omega^2 t}] \tag{30}$$

we see that

$$I(x, t) = I_0(x) * e^{-\frac{x^2}{2\sigma_t^2}} \tag{31}$$

where $\sigma_t = 2t$

This is Gaussian convolution

The Heat Equation / Diffusion Equation

- Evolving $I(x)$ according to the heat equation is equivalent to convolving with a Gaussian kernel.
- Longer evolution corresponds to convolution with Gaussians of higher variance.
- This gives us firm footing for studying PDEs in the context of image processing.

Outline

Next Class

Variational calculus.
Numerical methods for solving the heat equation.