

Medical Image Analysis

CS 593 / 791

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Outline

- 1 Discretizing the heat / diffusion equation
- 2 Matrix forms of the discretized heat equation
- 3 Scale space image representation

Outline

- 1 Discretizing the heat / diffusion equation
 - Numerical differentiation
 - First derivative approximations
 - Second derivative approximations
- 2 Matrix forms of the discretized heat equation
- 3 Scale space image representation

Introduction

The names 'heat equation' and 'diffusion equation' are used interchangeably: the same equation describes both phenomena.

$$\text{In 1D} \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\text{In 2D} \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\text{In 3D} \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\text{In general} \quad \frac{\partial u}{\partial t} = \text{div}(\nabla u)$$

- Heat : u is temperature
- Diffusion : u is concentration
- Images : u is image intensity

Introduction

- We know the heat equation can be solved analytically.

- ▶ $I(x, t) = I_0(x) * e^{-\frac{x^2}{2\sigma_t^2}}$

- In the future we will consider nonlinear variants of the diffusion equation for which no simple analytical solution exists.
- In these cases we approximate a solution numerically.

Introduction

Problem

Estimate derivatives of some unknown smooth function, $f(x)$, given only samples $\{f(x_i)\}$.

Motivation

Find approximate solutions to PDEs governing evolution of $f(x)$.

Forward Difference Equation

Expand $f(x)$ in a Taylor series about x_0 .

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + O(h)$$

Then evaluate at $x = x_0 + h$.

$$f(x_0 + h) = f(x_0) + hf'(x_0) + O(h)$$

First order forward difference:

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}$$

Backward Difference Equations

Replace h with $-h$ in the previous derivation

$$f(x_0 - h) = f(x_0) - hf'(x_0) + O(h)$$

First order backward difference:

$$f'(x_0) \approx \frac{f(x_0) - f(x_0 - h)}{h}$$

Centered Difference Equation

Subtract the first order expansions:

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \dots$$

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \dots$$

$$f(x_0 + h) - f(x_0 - h) = 2hf'(x_0)$$

Divide by $2h$ to get the centered difference:

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

Centered Difference Equation

The second order terms in the expansion cancel, so the remainder is $O(h^2)$

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + O(h^2)$$

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) + O(h^2)$$

For images : Use $h = 1$ pixel.

Second Centered Difference Equation

We can approximate the **second derivative** by adding two second order expansions:

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + O(h^2)$$

+

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) + O(h^2)$$

=

$$f(x_0 + h) + f(x_0 - h) = 2f(x_0) + h^2f''(x_0) + O(h^2)$$

Second Centered Difference Equation

Rearrange

$$f(x_0 + h) + f(x_0 - h) = 2f(x_0) + h^2 f''(x_0) + O(h^2)$$

to get the (second order) second centered difference

$$f''(x_0) = \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}$$

Outline

- 1 Discretizing the heat / diffusion equation
- 2 Matrix forms of the discretized heat equation
 - Forward difference method
 - Backward difference method
- 3 Scale space image representation

Explicit Method

Recall the heat equation

$$\frac{\partial I}{\partial t} = \frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2}$$

Use forward difference in time and second central difference in space:

$$\frac{I_{x,y}^{t+\delta} - I_{x,y}^t}{\delta} = (I_{x+1,y}^t - 2I_{x,y}^t + I_{x-1,y}^t) + (I_{x,y+1}^t - 2I_{x,y}^t + I_{x,y-1}^t)$$

- Superscripts : time
- Subscripts : position

Forming the linear system of equations

$$I_{x,y}^{t+\delta} = I_{x,y}^t + \delta(I_{x+1,y}^t - 4I_{x,y}^t + I_{x-1,y}^t + I_{x,y+1}^t + I_{x,y-1}^t)$$

- We want to simultaneously solve for $I_{x,y}^{t+\delta}$ at all x, y .
- The image can be stretched into a vector, \mathbf{w} , by stacking the columns of the image on top of each other.
 - ▶ Matlab : reshape command

$$I = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix} \rightarrow \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ 16 \end{bmatrix}$$

The evolution equation will be $\mathbf{w}^{t+\delta} = \mathbf{A}\mathbf{w}^t$

Reshaping image I into array \mathbf{w}

$$I = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix} \rightarrow \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ 16 \end{bmatrix}$$

allows us to rewrite the equations

$$I_{x,y}^{t+\delta} = I_{x,y}^t + \delta(I_{x+1,y}^t - 4I_{x,y}^t + I_{x-1,y}^t + I_{x,y+1}^t + I_{x,y-1}^t)$$

in terms of \mathbf{w} .

For example,

$$w_{10}^{t+\delta} = w_{10}^t + \delta(w_{14}^t - 4w_{10}^t + w_6^t + w_{11}^t + w_9^t)$$

this can be rewritten as a row vector times a column vector

$$w_{10}^{t+\delta} = [\dots \delta \dots \delta \quad 1 - 4\delta \quad \delta \dots \delta \dots] \mathbf{w}$$

Forming the linear system of equations

In general we have

$$w_i^{t+\delta} = w_i^t + \delta(w_{i+1}^t - 4w_i^t + w_{i-1}^t + w_{i+n}^t + w_{i-n}^t)$$

- Collect the coefficients of \mathbf{w} into a matrix, \mathbf{A} .
- If the image, $I(x, y)$ is size $n \times n$, the vector, \mathbf{w} , is $n^2 \times 1$,
- the matrix of coefficients, \mathbf{A} is $n^2 \times n^2$.
- Most elements of \mathbf{A} are 0 (i.e. \mathbf{A} is sparse)

Linear system of equations

$$w_i^{t+\delta} = w_i^t + \delta(w_{i+1}^t - 4w_i^t + w_{i-1}^t + w_{i+n}^t + w_{i-n}^t)$$

The matrix of coefficients

$$\mathbf{A} = \begin{pmatrix} (1 - 4\delta) & \delta & 0 & \dots & \delta & \dots & \dots & 0 & 0 \\ \delta & (1 - 4\delta) & \delta & 0 & \dots & \delta & \dots & \dots & 0 \\ 0 & \delta & (1 - 4\delta) & \delta & 0 & \dots & \delta & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Properties of \mathbf{A} :

- Symmetric,
- Sparse:
 - ▶ 5 nonzero diagonals for a 2D image
 - ▶ 7 nonzero diagonals for a 3D image
 - ▶ Matlab : Use `sparse`, `spdiags`

Explicit or forward solution

Each iteration is a matrix multiplication.

$$\mathbf{w}^{t+\delta} = \mathbf{A}\mathbf{w}^t$$

Convergence Criterion

Steady-state is reached when $\mathbf{w}^{t+\delta} \approx \mathbf{w}^t$.

Check $\|\mathbf{w}^{t+\delta} - \mathbf{w}^t\| < \epsilon$

Problem

- For large δ we may overshoot the solution.
- The iteration will oscillate, and never converge.
- Stability is only guaranteed for small δ , and then convergence is slow.

Implicit Method

Recall the heat equation

$$\frac{\partial I}{\partial t} = \frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2}$$

Use **backward difference** in time and second central difference in space:

$$\frac{I_{x,y}^t - I_{x,y}^{t-\delta}}{\delta} = (I_{x+1,y}^t - 2I_{x,y}^t + I_{x-1,y}^t) + (I_{x,y+1}^t - 2I_{x,y}^t + I_{x,y-1}^t)$$

Linear system for backward difference method

The generic difference equation

$$I_{x,y}^{t-\delta} = I_{x,y}^t - \delta(I_{x+1,y}^t - 4I_{x,y}^t + I_{x-1,y}^t + I_{x,y+1}^t + I_{x,y-1}^t)$$

has the vector form

$$w_i^{t-\delta} = w_i^t - \delta(w_{i+1}^t - 4w_i^t + w_{i-1}^t + w_{i+n}^t + w_{i-n}^t)$$

Linear system of equations

$$\mathbf{w}^{t-\delta} = \mathbf{B}\mathbf{w}^t$$

The matrix of coefficients

$$\mathbf{B} = \begin{pmatrix} (1 + 4\delta) & -\delta & 0 & \dots & -\delta & \dots & \dots & 0 & 0 \\ -\delta & (1 + 4\delta) & -\delta & 0 & \dots & -\delta & \dots & \dots & 0 \\ 0 & -\delta & (1 + 4\delta) & -\delta & 0 & \dots & -\delta & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \ddots & \end{pmatrix}$$

Properties of \mathbf{B} :

- Symmetric,
- Sparse:
 - ▶ 5 nonzero diagonals for a 2D image
 - ▶ 7 nonzero diagonals for a 3D image

Implicit or backward solution

Each iteration requires solution of a linear system (inversion or factorization).

$$\mathbf{B}\mathbf{w}^t = \mathbf{w}^{t-\delta}$$

This method is stable, however setting δ too large will result in slow convergence.

Matrix stability analysis

If we have some small error, \mathbf{e}^0 in the initial condition \mathbf{w}^0

$$\mathbf{w}^1 = \mathbf{A}(\mathbf{w}^0 + \mathbf{e}^0) = \mathbf{A}\mathbf{w}^0 + \mathbf{A}\mathbf{e}^0$$

and at the next iteration we have

$$\mathbf{w}^2 = \mathbf{A}(\mathbf{A}\mathbf{w}^0 + \mathbf{A}\mathbf{e}^0) = \mathbf{A}^2\mathbf{w}^0 + \mathbf{A}^2\mathbf{e}^0.$$

In general, at iteration n , we have

$$\mathbf{w}^n = \mathbf{A}^n\mathbf{w}^0 + \mathbf{A}^n\mathbf{e}^0.$$

Whether $\|\mathbf{A}^n\mathbf{e}^0\| \leq \|\mathbf{e}^0\|$ depends on the **condition number** of matrix \mathbf{A} .

Condition number(\mathbf{A}) ≈ 1 is well-conditioned.

Matrix stability analysis

Computing the condition number is difficult, but...

As a general rule:

Strictly diagonally dominant matrices are well-conditioned.

Definition : A matrix, A , is **strictly diagonally dominant** if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

for all rows i .

Stability of the forward difference equation

A typical row of matrix \mathbf{A} :

$$\mathbf{A}_i = (0 \quad \dots \quad \delta \quad 0 \quad \dots \quad \delta \quad (1 - 4\delta) \quad \delta \quad 0 \quad \dots \quad \delta \quad 0 \quad \dots)$$

For what values of δ is the matrix diagonally dominant?

$$|1 - 4\delta| > 4|\delta|$$

$$\frac{1 - 4\delta}{4\delta} > 1 \rightarrow 1 - 4\delta > 4\delta \rightarrow 1 > 8\delta$$

or

$$\frac{1 - 4\delta}{4\delta} < -1 \rightarrow 1 - 4\delta < -4\delta \rightarrow 1 < 0$$

The forward difference method is stable when $\delta < \frac{1}{8}$.

Stability of the backward difference equation

A typical row of matrix \mathbf{B} :

$$\mathbf{B}_i = (0 \quad \dots \quad -\delta \quad 0 \quad \dots \quad -\delta \quad (1 + 4\delta) \quad -\delta \quad 0 \quad \dots \quad -\delta \quad 0 \quad \dots)$$

For what values of δ is the matrix diagonally dominant?

$$|1 + 4\delta| > 4|\delta|$$

$$\frac{1 + 4\delta}{4\delta} > 1 \rightarrow 1 + 4\delta > 4\delta \rightarrow 1 > 0$$

or

$$\frac{1 + 4\delta}{4\delta} < -1$$

The backward difference method is stable for all δ . (Unconditionally stable)

Matrix stability analysis

For more details about matrix condition number and spectral radius, see *Golub and Van Loan*, "**Matrix Computations**" or another numerical linear algebra text.

In Matlab use `cond`, `condest`

Implicit or backward solution

Each iteration requires solution of a linear system (inversion or factorization).

$$\mathbf{B}\mathbf{w}^t = \mathbf{w}^{t-\delta}$$

This method is unconditionally stable, however setting δ too large will result in slow convergence.

Problem

Error is of order $O(\delta)$.

Mixed Explicit/Implicit Method

We can get order $O(\delta^2)$ error by averaging the two difference equations. Note that the second order terms in the Taylor series cancel, just like they did when we computed central differences.

$$\begin{aligned} & \frac{1}{2} \mathbf{w}^{t+\delta} = \frac{1}{2} \mathbf{A} \mathbf{w}^t \\ + & \\ & \frac{1}{2} \mathbf{B} \mathbf{w}^{t+\delta} = \frac{1}{2} \mathbf{w}^t \\ = & \\ & (\mathbf{B} + \mathbf{I}) \mathbf{w}^{t+\delta} = (\mathbf{A} + \mathbf{I}) \mathbf{w}^t \end{aligned}$$

Solutions to the linear system

$$(\mathbf{B} + \mathbf{I})\mathbf{w}^{t+\delta} = (\mathbf{A} + \mathbf{I})\mathbf{w}^t$$

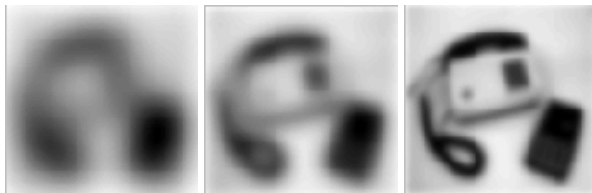
- Don't try to invert the matrix $\mathbf{B} + \mathbf{I}$.
- Instead use Gaussian elimination, LU decomposition.

Outline

- 1 Discretizing the heat / diffusion equation
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- 3 Scale space image representation
 - The concept
 - A requirement
 - Next Class

What is scale space?

We may be interested in image features which exist at different scales in the image, so we want to represent an image over a continuum of scales (coarse to fine).



What is scale space?

We may be interested in image features which exist at different scales in the image, so we want to represent an image over a continuum of scales (coarse to fine).



Scale-space requirement

As we progress from fine to coarse through scale space, we should not create new details.

- Images generated by isotropic diffusion satisfy this requirement.
- Equivalent to convolution / low-pass filtering
- However, edges at the coarse scales are blurred.
- We must track features up to the finest scale to get their true locations.



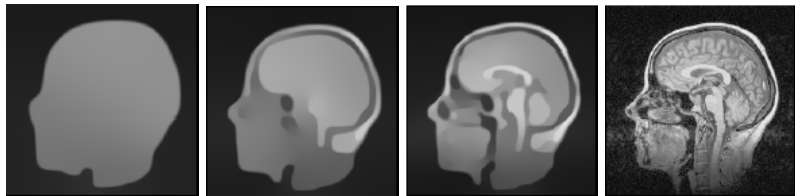
New scale space representation

Perona and Malik suggest a new technique for generating the scale space of images which preserves edges in "*Scale-Space and Edge Detection Using Anisotropic Diffusion*". They propose the use of

- **Inhomogeneous diffusion:** rate of diffusion varies spatially.

Weickert, in "*A Review of Nonlinear Diffusion Filtering*.", proposes

- **Anisotropic diffusion:** rate of diffusion at a point varies with direction.
to generate the scale space of images and perform denoising.



Wednesday (MLK B'day is Monday)

- Quiz about the reading.
- The physical process of diffusion.
- Discuss the 2 papers.