Medical Image Analysis

CS 593 / 791

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Outline

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1. Coordinate Transformations
   - Global Transformations
   - Local Transformations

2. Principal Axes Transformation

Problem Definition

Image registration is the process of determining a coordinate transformation between two images that are misaligned.

$$\min_{T} dist(I_1(\mathbf{x}), I_2(T(\mathbf{x})))$$

- T is a coordinate transformation
- $I_1(\mathbf{x})$ and $I_2(\mathbf{x})$ are 2 images to be aligned
- $dist(I_1, I_2)$ is a metric which determines how well the images match.
- $dist(I_1, I_2)$ can be based on image intensities or extracted features.

Linear transformations

- Translation
- Rotation
- Scaling
- Shear

2D Linear transformations

Translation : 2 parameters

$$T(\mathbf{x}) = \mathbf{x} + \mathbf{t}$$

Rotation about the origin : 1 parameter

$$T(\mathbf{x}) = R_z\mathbf{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{x}$$

Nonuniform scaling : 2 parameters

$$T(\mathbf{x}) = S\mathbf{x} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \mathbf{x}$$

Shear : 1 parameter

$$T(\mathbf{x}) = C\mathbf{x} = \begin{bmatrix} 1 & \cot\theta \\ 0 & 1 \end{bmatrix} \mathbf{x}$$

2D Linear transformations

A more general 2D transformation can be obtained by composing several transformations, such as:

$$T(\mathbf{x}) = R_z S(\mathbf{x} - \mathbf{c}) + \mathbf{t}$$

- Translate so that center of the rotation is [0,0].
- Scale the coordinate systems.
- Rotate about the origin.
- Translate.
- Total of 7 parameters.

3D Linear transformations

- Translation : 3 parameters
- Scale : 3 parameters
- Shear : 2 parameters
- Rotation : 3 Euler angles

Using Euler angles, the 3D rotation is represented as 3 consecutive rotations about the coordinate axes.

$$T(\mathbf{x}) = R_x R_y R_z \mathbf{x}$$

where $R_x R_y R_z =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation matrices

Rotation by θ about the origin is represented by the matrix

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

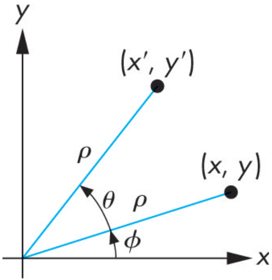
Rotation matrices are orthogonal : $R^{-1} = R^T$.

$$R_z(\theta)^{-1} = R_z(-\theta) = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

Since cos is an even function $\cos(-\theta) = \cos(\theta)$
and sin is an odd function $\sin(-\theta) = -\sin(\theta)$,

$$R_z(-\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R_z(\theta)^T$$

2D Rotation : Geometric derivation



Rewrite in polar coordinates:

$$
\begin{aligned}
x &= \rho \cos \phi \\
y &= \rho \sin \phi \\
x' &= \rho \cos(\theta + \phi) \\
y' &= \rho \sin(\theta + \phi)
\end{aligned}
$$

Using the trig identities

$$
\begin{aligned}
\cos(\theta + \phi) &= \cos \phi \cos \theta - \sin \phi \sin \theta \\
\sin(\theta + \phi) &= \cos \phi \sin \theta + \sin \phi \cos \theta
\end{aligned}
$$

Rewrite x', y' in terms of x, y

$$
\begin{aligned}
x' &= \rho \cos \phi \cos \theta - \rho \sin \phi \sin \theta = x \cos \theta - y \sin \theta \\
y' &= \rho \cos \phi \sin \theta + \rho \sin \phi \cos \theta = x \sin \theta + y \cos \theta
\end{aligned}
$$

2D Rotation : Geometric derivation

We can rewrite this system of equations

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned}$$

in matrix form as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

This is equivalent to 3D rotation about the z-axis

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Displacement Field

Compute a displacement vector for each voxel.

$$T(\mathbf{x}) = \mathbf{x} + \mathbf{t}(\mathbf{x})$$

To constrain the displacement field to represent physically plausible deformations, we may impose smoothness constraints.

If $\mathbf{t}(\mathbf{x}) = [u(x,y), v(x,y)]$,

$$\min_{u,v} \int \int ((u_x^2 + u_y^2) + (v_x^2 + v_y^2)) dx\, dy$$

will constrain the displacement field to be smooth

Displacement Field

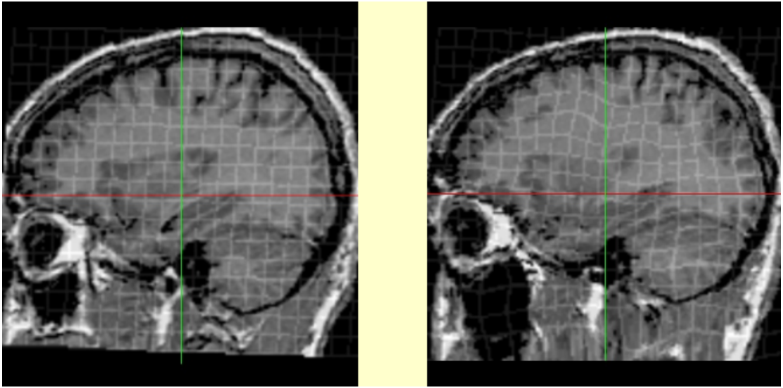
Viscoelastic regularization methods:

- Consider the deformation field to be the velocity field of some viscous fluid.
- More suitable for large deformations
- Constrain the field to obey the Navier-Stokes equation.
- Smoothness of the field is controlled by the viscosity of the simulated fluid.
- Computationally expensive

Spline based transformations

- Fewer control points than image pixels.
- The spline may interpolate or approximate the control points.
- Sum of shifted basis functions.
- Basis functions may have local or global support.
- Basis functions are generally low degree (3) polynomials.

Spline based transformations



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   - Local Transformations
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Principal Axes Transformation

Characterize images by

- Centroid (a 2D or 3D point)
- Principal directions (2 or 3 perpendicular vectors)

Images I_1 and I_2 can be aligned by

- Translating centroid 2 to be coincident with centroid 1
- Rotate about centroid 1 so that the principal directions are aligned.

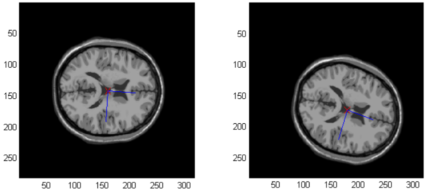

Image centroid

The expected value of a function, f, with respect to another function $B(x)$ is defined as

$$E_B[f] = \frac{\int_\Omega f(x)B(x)dx}{\int_\Omega B(x)dx}.$$

The centroid, $\mathbf{c_I}$, of the image, I($\mathbf{x}$) is $E_I[\mathbf{x}]$.

- $\mathbf{c_I}$ is the center of the distribution of image intensities.
- $\mathbf{c_{I_1}}$ and $\mathbf{c_{I_2}}$ should be corresponding points in the 2 images.

Image principal axes

The covariance, cov_I of the image, I($\mathbf{x}$) is $E_I[(\mathbf{x}-\mathbf{c_I})(\mathbf{x}-\mathbf{c_I})^T]$.

$$cov_I = \Sigma\Lambda\Sigma^T$$

where

- Λ is a diagonal matrix (scaling) describing the variation of I in the principal directions.
- Σ is an orthogonal matrix (rotation) which rotates the coordinate axes onto the principal directions.
- The columns of Σ are the principal directions.

So if

- $cov_{I_1} = \Sigma_1\Lambda_1\Sigma_1{}^T$
- $cov_{I_2} = \Sigma_2\Lambda_2\Sigma_2{}^T$

then the rotation which aligns the axes of I_2 with the axes of I_1 is $\Sigma_1\Sigma_2^T$