Medical Image Analysis

CS 593 / 791

Computer Science and Electrical Engineering Dept.
West Virginia University

26th January 2007

Outline

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Definition

The TV-norm is the L1 functional norm of the gradient magnitude.

$$TV(u) = \int_{\Omega} ||\nabla u|| \, dx \, dy$$

Recall

Minimizing the L2 functional norm of the gradient magnitude led to the isotropic heat equation.

Membrane Spline vs Total Variation

The **membrane spline** energy functional

$$\int_\Omega ||\nabla u||^2 \, dx \, dy$$

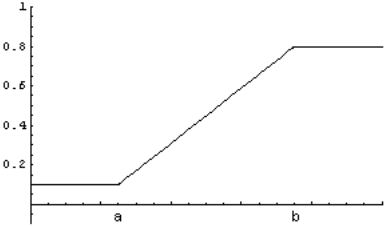
represents the elastic potential energy of a thin sheet of material.
The **total variation** energy functional

$$\int_\Omega ||\nabla u|| \, dx \, dy$$

represents the oscillation of the function u.

Total Variation Does Not Penalize Discontinuities

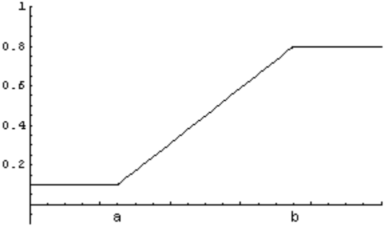
Consider a function, u, with a step of height h:



$$\begin{aligned} MEM(f) &= \int_\Omega u_x^2\, dx = \int_a^b \left(\frac{h}{b-a}\right)^2 dx \\ &= \left(\frac{h}{b-a}\right)^2 \int_a^b dx = \left(\frac{h}{b-a}\right)^2 (b-a) \\ &= \frac{h^2}{b-a} \end{aligned}$$

Total Variation Does Not Penalize Discontinuities

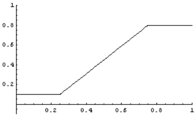
Consider a function, u, with a step of height h:



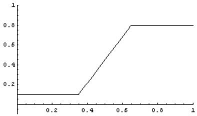
$$
\begin{aligned}
TV(f) &= \int_\Omega \sqrt{u_x^2}\, dx = \int_a^b \left|\frac{h}{b-a}\right| dx \\
&= \left|\frac{h}{b-a}\right| \int_a^b dx = \left|\frac{h}{b-a}\right| (b-a) \\
&= |h|
\end{aligned}
$$

Total Variation Does Not Penalize Discontinuities

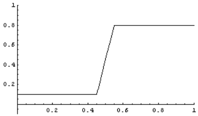
$$MEM(f) = \frac{h^2}{b-a},\ TV(f) = |h|$$



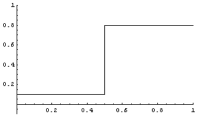
MEM(f1) = 0.98
TV(f1) = 0.7



MEM(f2) = 1.63
TV(f2) = 0.7

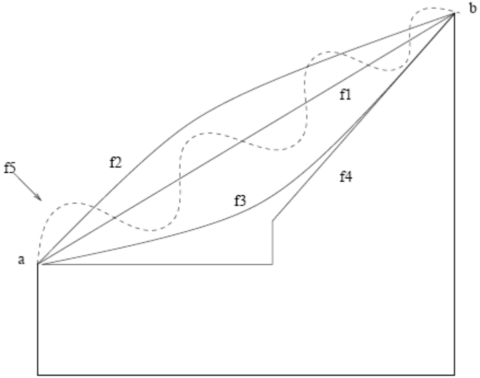


MEM(f3) = 4.9
TV(f3) = 0.7



MEM(f4) = ∞
TV(f4) = 0.7

Total Variation Does Penalize Oscillation



$$TV(f_5) > TV(f_1) = TV(f_2) = TV(f_3) = TV(f_4)$$

Outline

Variational Calculus

Specifically, minimizing the membrane spline energy

$$E(u) = \int_\Omega ||\nabla u||\, dx = \int_\Omega \sqrt{u_x^2 + u_y^2}\, dx$$

has Euler-Lagrange condition for minimization:

$$\begin{aligned} \nabla E &= f_u - \frac{\partial}{\partial x} f_{u_x} - \frac{\partial}{\partial y} f_{u_y} = 0 \\ &= 0 - \frac{\partial}{\partial x}(u_x(u_x^2 + u_y^2)^{-\frac{1}{2}}) - \frac{\partial}{\partial y}(u_y(u_x^2 + u_y^2)^{-\frac{1}{2}}) = 0 \end{aligned}$$

Leading to the descent equation

$$\partial_t u = \mathrm{div}(\frac{\nabla u}{||\nabla u||})$$

Energy Minimization

Membrane spline energy

$$\min_u \int_\Omega u_x^2 + u_y^2 \, dx \, dy$$

Has Euler-Lagrange condition for minimization given by

$$\text{div}(\nabla u) = 0$$

and descent equation

$$\partial_t u = \text{div}(\nabla u)$$

Total Variation

$$\min_u \int_\Omega \sqrt{u_x^2 + u_y^2} \, dx \, dy$$

Has Euler-Lagrange condition for minimization given by

$$\text{div}\left(\frac{\nabla u}{||\nabla u||}\right) = 0$$

and descent equation

$$\partial_t u = \text{div}\left(\frac{\nabla u}{||\nabla u||}\right)$$

Descent Equation

$$\partial_t u = \text{div}\left(\frac{\nabla u}{||\nabla u||}\right)$$

This is an inhomogeneous diffusion equation with diffusivity

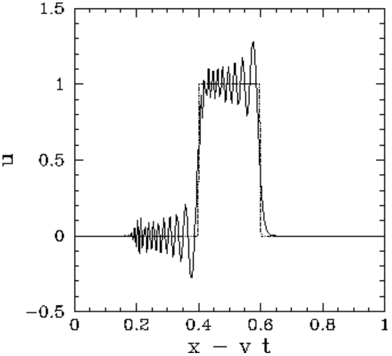
$$g(||\nabla u||) = \frac{1}{||\nabla u||}$$

- Diffusion is slow for large image gradients.
- Diffusion is fast for small image gradients.
- Divergence is due only to changing gradient directions.

Outline

Upwind finite differences

Since we are allowing steep discontinuities, numerical differentiation can be a source of instability.



Solution : a slope limiting derivative operator.

Denote the forward finite difference of u at (i, j) in the x direction by:

$$\Delta_+^x u_{ij} = u_{i+1,j} - u_{i,j}$$

and the backward difference by:

$$\Delta_-^x u_{ij} = u_{i,j} - u_{i-1,j}$$

We will denote the upwind finite difference in the x direction as:

$$\text{minmod}(\Delta_+^x u_{ij}, \Delta_-^x u_{ij})$$

Minmod operation

$$\operatorname{minmod}(a,b) = \left(\frac{\operatorname{sgn} a + \operatorname{sgn} b}{2}\right) \min(|a|,|b|)$$

where

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ +1 & \text{if } x > 0. \end{cases}$$

Minmod operation

$$\mathrm{minmod}(a,b) = \left(\frac{\mathrm{sgn}\, a + \mathrm{sgn}\, b}{2}\right) \min(|a|,|b|)$$

- if a or b are 0, $\mathrm{minmod}(a,b) = 0$
- if a and b have opposite sign, $\mathrm{minmod}(a,b) = 0$
- otherwise, pick the one with the smallest magnitude.

What happens when $||\nabla u|| \to 0$

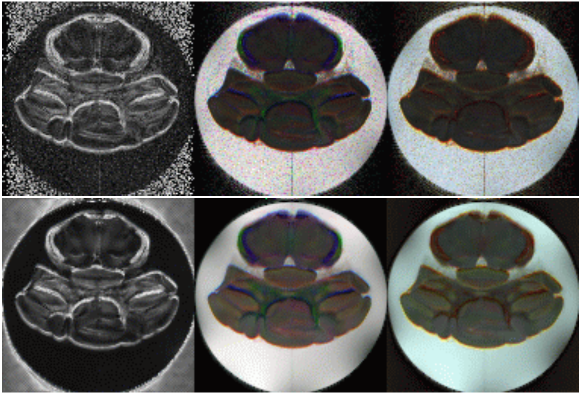
Singularities may develop in very smooth regions.

$$\partial_t u = \text{div}(\frac{\nabla u}{||\nabla u|| + \epsilon})$$

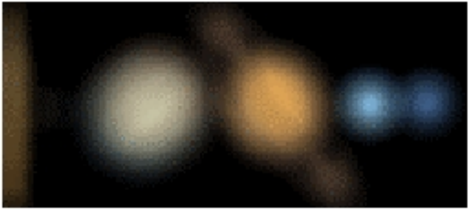
- One strategy: use fixed constant $\epsilon \ll 1$.
- Another strategy: relaxation. Start with a moderate ϵ and decrease over time.

Outline

Denoising Results



Deconvolution Results



A Drawback

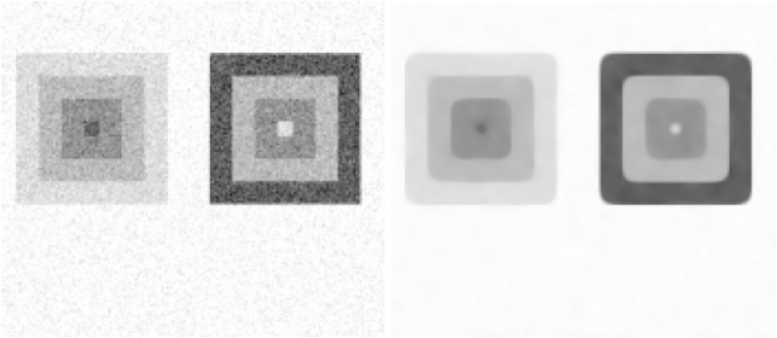
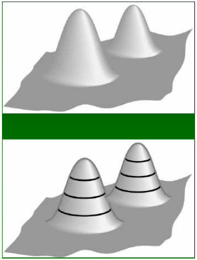


Image contours may be rounded.

Image Contours

Another geometric interpretation of TV norm minimization: Consider isocontours of the image (the curves of constant image intensity)



- Evolution of the image u is also evolution of the isocontours, c.
- TV-norm minimization is smoothing of the isocontours of u.