1 Sets

A *set* is a collection of objects that share some common property. Any object that has the property is a member of the set, and any object that does not have the property is not a member. Sets can be finite or infinite. We call members of a set *elements*.
Notation. We use capital letters to denote sets and the symbol $\in$ to denote membership in a set. $a \in A$—a is a member (or an elment) of set A.
There are 3 ways to describe a set

1. List (or partially list) its elements. For example, Bridge Club= {Mary, Bob, Alice, Joe, Amy}.

   Note that no ordering is imposed on the elements of a set. So {Bod, Alice, Joe, Mary, Amy} denotes the same set (Two sets are *equal* if they contain the same elements). Also, each member of a set is listed only once; multiple listings of a single element will be counted only once anyway.

2. Use recursion to describe how to generate the set elements. For example,

   1. $2 \in S$
   2. If $n \in S$, then $(n+2) \in S$

3. Describe a property P that characterizes the set elements. For example,

   $$E = \{x | x \in \mathbb{Z},\ x \geq 0 \text{ and } x \text{ is even}\}.$$

   This method is most commonly used. The notation for a set whose elements are characterized as having property P is $\{x|P(x)\}$. Property P is called the defining property for the set.

   The above example is usually written as

   $$E = \{x \in \mathbb{Z} | x \geq 0 \text{ and } x \text{ is even}\}$$

   where $x \in \mathbb{Z}$ is the restrict domain for set E.

Familiar Sets.
$\mathbb{N}$—set of all nonnegative integers, i.e., $\{0, 1, 2, 3, \ldots\}$.
$\mathbb{Z}$—set of all integers, i.e., $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
$\mathbb{Q}$—set of all rational numbers, i.e., only numbers that can be expressed as a ratio $\frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$.
$\mathbb{R}$—set of all real numbers.
$\mathbb{C}$—set of all complex numbers.
$\emptyset$—set with no elements. (the empty/null set)

Relationship Between Sets
A is a *subset* of B if every member of A is also a member of B , denoted by $A \subseteq B$.

If $A \subseteq B$ but $A \neq B$, then A is a *proper subset* of B, denoted by $A \subset B$. For example, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.
Table of formal logic notations of sets relationship

Set Relationship	Formal Logic Notation
$A = B$	$(\forall x)[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A]$
$A \subseteq B$	$(\forall x)(x \in A \rightarrow x \in B)$
$A \subset B$	$(\forall x)(x \in A \rightarrow x \in B) \wedge (\exists y)(y \in B \wedge y \notin A)$
$A \not\subseteq B$	$(\exists x)(x \in A \wedge x \notin B)$

Power Set of a Set
For a set S, we can form a new set whose elements are all of the subsets of S. This new set is called the *power set* of S, $\mathcal{P}(S)$. For example, given $A = \{a, b, c\}$, $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. If S has size k, then $\mathcal{P}(S)$ has size 2^k.

Operations on Sets Let S be the universal set, the set of objects being discussed. Let $A, B \in \mathcal{P}(S)$.
The *union* of A and B, $A \cup B$, is $\{x \in S | x \in A \text{ or } x \in B\}$, the set of all things that are either in A or in B.
The *intersection* of A and B, $A \cap B$, is $\{x \in S | x \in A \text{ and } x \in B\}$, the set of all things that are both in A and in B.
The *complement* of A, A', is $\{x \in S | x \notin S\}$, the set of all things that are not in A.
The above operations can be illustrated in Venn Diagrams.

The *difference* of A and B, $A - B$, is $\{x \in S | x \in A \text{ and } x \notin B\}$, the set of all things that are in A but not in B.
The *Cartesian product* (or *cross product*) of A and B, $A \times B$, is $\{(x, y) | x \in A \text{ and } x \in B\}$, the set of all ordered pairs whose first component comes from A and whose second component comes from B. A^2 means $A \times A$.

Binary and Unary Operations
Definition: $\circ$ is a *binary operation* on a set S if for every ordered pair (x, y) of elements of S, $x \circ y$ exists, is unique, and is a member of S.

That the value $x \circ y$ always exists and is unique is described by saying that the binary operation $\circ$ is *well-defined*. The property that $x \circ y$ always belongs to S is described by saying that S is *closed* under operation $\circ$.

Example 1.1. *1. Addition, subtraction, and multiplication are all binary operations on on* $\mathbb{Z}$.
2. The logical operations of $\wedge$, $\vee$, $\rightarrow$ *and* $\leftrightarrow$ *are binary operations on the set of propositional wffs.*
3. Division is not a binary operation on $\mathbb{Z}$ *because* $x \div 0$ *is not defined.*
4. Subtraction is not a binary operation on $\mathbb{N}$ *because* $\mathbb{N}$ *is not closed under subtraction.*
5. $\cup$, $\cap$, *and* $-$ *are binary operations on any universal set* S.
6. Cartisian product $\times$ *is not a binary operation.*

Set Identities

BASIC SET IDENTITIES

1a.	$A \cup B = B \cup A$	1b.	$A \cap B = B \cap A$	commutative
2a.	$(A \cup B) \cup C = A \cup (B \cup C)$	2b.	$(A \cap B) \cap C = A \cap (B \cap C)$	associative
3a.	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	3b.	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	distributive
4a.	$A \cup \emptyset = A$	4b.	$A \cup S = A$	identity
5a.	$A \cup A' = S$	4b.	$A \cup A' = \emptyset$	complement

Set identities can be derived by drawing a Venn diagram, establishing set inclusion in each direction, or using already proved identities. See examples in the book.

2 Counting

Multiplication Principle

If there are n_1 possible outcomes for the first event and n_2 possible outcomes for a second event, there are $n_1 \cdot n_2$ possible outcomes for the sequence of the two events.

Remark: Multiplication principle applies when the # of outcomes from on event is independent of the outcome of the other event.

Example 2.1. *1. The possible number of telephone numbers, assuming 10-digit phone number.*
2. How many ways are there to choose three officers from a club of 25 people?
3. The order of $A \times B$ *is* $|A| \times |B|$.

Addition Principle

If A and B are disjoint events with n_1 and n_2 possible outcomes, respectively, then the total number of possible outcomes from the event "A or B

Example 2.2. *1. A customer wants to purchase a vehicle from a dealer. The dealer has 23 autos and 14 trucks in stock. How many selections does the customer have?*
2. $|A - B| = |A| - |A \cap B|$.

We can have complex examples by using two principles together.

Example 2.3. *1. How many four-digit numbers begin with a 4 or a 5?*
2. How many three-digit integers are even?
3. How many four-digit numbers are there in which 3 is not repeated?

3 Principle of Inclusion and Exclusion; Pigeonhole Principle

We denote by $|A|$ the number of members in A (the order of A). The Principle of Inclusion and Exclusion states that

$$|A \cup B| = |A| + |B| - |A \cap B| \tag{1}$$

and

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \tag{2}$$

Example 3.1. *A survey of 30 people indicates that 25 of them like football or basket ball, 8 of them like basketball only, and 5 of them like both. How many of them like football only?*

Example 3.2. *How many 8 character bit strings are there that begin or end with 1?*

Example 3.3. *A produce stand sells only broccoli, carrots, and okra. One day the stand served 207 people. If 114 people purchased broccoli, 152 purchased carrots, 25 purchased okra, 64 purchased broccoli and carrots, 12 purchased carrots and okra, and 9 purchased all three, how many people purchased broccoli and okra?*

The general principle is as follows.

Principle of Inclusion and Exclusion

Given the finite sets $A_1, \cdots, A_n$, $n \geq 2$, then

$$\begin{aligned}|A_1 \cup \cdots A_n| &= \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \\ &+ \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| \\ &- \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n| \end{aligned} \tag{3}$$

Pigeonhole Principle

If more than k items are placed into k bins, then at least one bin contains more than one items.

Example 3.4. *1. How many people must be in a room to gurantee that two people have last names that begin with the same initial?*
2. How many times must a single die be rolled in order to guarantee getting the same value twice?
3. Prove that if 51 positive integers between 11 and 100 are chosen, then one of them must divide another.

4 Permutations and Combinatorics

Permutation Any ordered arrangement of objects is called a *permutation*. The number of permutations (order matters) of r distinct objects chosen from n distinct objects is denoted by $P(n,r)$. Let $n! = n(n-1)\cdots 1$, then

$$P(n,r) = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

$0! = 1$ by default.

Example 4.1. *1. How many 4 digit sequences are there using any digit without repeating a digit?*
2. How many strings can one get by choosing 3 distinct characters out of $\{a,b,c,d,e\}$*?*
3. 10 athletes compete in an Olympic event. Gold, silver, and bronze medals are awarded; in how many ways can the awards be made?
4. A library has 4 books on operating systems, 7 on programming, and 3 on data structures. how many ways these books can be arranged on a shelf, given that all books on the same subject must be together?

Combination
Q: Write down the ways of chooseing 3 people from a family of 5 to go Christmas shopping.
A:
The number of combinations (order does not matter) of r distinct objects chosen from n distinct objects is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

Note that $C(n,0) = C(n,n) = 1$, $C(n,1) = C(n,n-1) = n$, ..., $C(n,r) = C(n,n-r)$.

Example 4.2. *How many 5-card poker hands can be dealt from a 52-card deck?*

Example 4.3. *A committee of 8 students is to be selected from a class consisting of 19 freshmen and 34 sophomores.*
a. In how many ways can 3 freshmen and 5 sophomores be selected?
b. In how many ways can a committee with exactly 1 freshman be selected?
c. In how many ways can a committee with at most 1 freshman be selected?
d. In how many ways can a committee with at least 1 freshman be selected?

Eliminating Duplicates

Example 4.4. *In Example 4.3, consider the number of ways to select a committee with at least 1 freshman. A bogus solution would be* $C(19,1)\cdot C(52,7)$ *[guarantee a freshman, then choose the rest 7 without restriction]. For a feasible set of committee candidates* S *that has two freshmen* A *and* B*,* S *was counted twice when* A *or* B *were chosen as the guaranteed freshman respectively.*

Another example is to find out the number of permutations for n_1 1's, n_2 2's, ..., and n_r r's. The solution is

$$\frac{(n_1 + n_2 + \cdots n_r)!}{n_1!n_2!\cdots n_r!}$$

Example 4.5. *a. How many distinct permutations can be made from the characters in the word **FLORIDA**?*
*b. How many distinct permutations can be made from the characters in the word **MISSISSIPPI**?*

Permutations and Combinations with Repetitions
The number of permutaions of r objects out of n distinct objects with repetition allowed is n^r.
The number of combinations of r objects out of n distinct objects with repetition allowed is $C(r+n-1, r)$.

Example 4.6. *A jeweler designing a pin has decided to use five stones chosen from diamonds, rubies, and emeralds. In how many ways can the stones be selected?*

Example 4.7. *How many distinct nonnegative integer solutions are there to the equation* $x_1 + x_2 + x_3 + x_4 = 10$*?*

Double Counting and Combinatorial Identities Two combinatorial formulas are equal if they count the size of the same set of objects.

Example 4.8. *1. Prove that* $C(n,r)C(r,k) = C(n,k)C(n-k, r-k)$ *for any* $r \leq n$ *and* $k \leq r$.
2. Prove Vandermonde's identity:

$$C(n+m, r) = \sum_{k=0}^{r} C(n,k)C(m, r-k)$$