Iterated function systems and the global construction of fractals

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Iterated function systems (i.f.s.s) are introduced as a unified way of generating a broad class of fractals. These fractals are often attractors for i.f.s.s and occur as the supports of probability measures associated with functional equations. The existence of certain \( p \)-balanced measures for i.f.s.s is established, and these measures are uniquely characterized for hyperbolic i.f.s.s. The Hausdorff–Besicovitch dimension for some attractors of hyperbolic i.f.s.s is estimated with the aid of \( p \)-balanced measures. What appears to be the broadest framework for the exactly computable moment theory of \( p \)-balanced measures – that of linear i.f.s.s and of probabilistic mixtures of iterated Riemann surfaces – is presented. This extensively generalizes earlier work on orthogonal polynomials on Julia sets. An example is given of fractal reconstruction with the use of linear i.f.s.s and moment theory.

1. Introduction

1.1. Preliminary remarks

We introduce iterated functions systems (i.f.s.s) first as a unified way of generating and classifying a broad class of fractals (Mandelbrot 1982) which contains classical Cantor sets, dragon curves, limit sets of Kleinian groups, Sierpinski gaskets, Julia sets, and much more. Many of these sets are traditionally viewed as being produced by a process of successive microscopic refinement taken to the limit. Our view is that they are attractors of i.f.s.s (explained in §§1.2 and 1.3) and as such can be generated globally. Plotted points all lie on the ‘fully refined’ fractal and distribute themselves probabilistically.

What is a fractal? Mandelbrot formally defines it to be a set whose Hausdorff–Besicovitch dimension exceeds its topological dimension. The spirit of the fractals we discuss is that, in one way or another, they are built up by overlaying smaller copies of themselves. We find it most natural to think of these fractals in terms of probability measures associated with functional equations: fractals occur as supports of fractal measures. To this end, we introduce in §2 the theory of \( p \)-balanced measures associated with i.f.s.s. These generalize the balanced measures that are associated with Julia sets (Brolin 1965; Barnsley et al. 1982; Bessis et al. 1982). We prove the existence of these measures and establish their relation with recurrent sets in §2.1; for hyperbolic i.f.s.s, studied in §2.2, we characterize these

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measures in terms of Daniel–Kolmogorov measures for Bernoulli systems. In §2.3 we study a special case of hyperbolic i.f.s., associated with dynamical systems, and obtain upper and lower bounds on the Hausdorff–Besicovitch dimension of the associated fractal with the aid of $p$-balanced measures.

In §3 we consider the moment theory of $p$-balanced measures for linear i.f.s and for an i.R.s. (iterated Riemann surface), which is a special class of i.f.s. We present what is apparently the broadest framework within which the moments of $p$-balanced measures can be computed exactly. This extensively generalizes the results of Pitcher & Kinney (1968), Barnsley et al. (1982), Bessis et al. (1982), Bessis et al. (1985), Barnsley & Harrington (1984) and Barnsley & Demko (1983) concerning the orthogonal polynomials and moment theory of probabilistic mixtures of Julia sets. The computability of moments is the key to our new approximation theory (Barnsley & Demko 1983): given the moments and other information such as the structure of an associated functional equation for a fractal measure $\mu$, one can find a sequence of approximate measures which make good use of the given information and converge in a satisfactory manner to $\mu$. We conclude §3 with an example of fractal reconstruction with the use of linear i.f.s and moment theory.

1.2. Iterated function systems

Let $K$ be a compact metric space and

$$\mathbf{w} = \{ w_i : i = 1, 2, \ldots, d \}$$

be a finite collection of Borel measurable functions $w_i : K \to K$. Let $C(K)$ denote the Banach space of continuous real-valued functions on $K$, with norm

$$\| f \|_\infty = \max \{ |f(x)| : x \in K \}, \quad f \in C(K).$$

We use the notation

$$\mathbf{p} = \{ p_i : i = 1, 2, \ldots, d \}$$

for a set of probabilities $p_i \geq 0$, with

$$\sum_{i=1}^{d} p_i = 1.$$ 

We will write $\mathbf{p} > 0$ to mean $p_i > 0$ for each $i$.

Definition 1. $\{K, \mathbf{w}\}$ is an iterated function system (i.f.s.) if and only if there is an associated set of probabilities $\mathbf{p}$ such that the operator $T$ on $C(K)$, given by

$$(Tf)(x) = \sum_{i=1}^{d} p_i (f \circ w_i)(x) \quad \text{for} \quad f \in C(K),$$

has the property

$$T : C(K) \to C(K).$$

If each $w_i$ is continuous then $\{K, \mathbf{w}\}$ is an i.f.s. and any set of probabilities $\mathbf{p}$ can be associated with it. The motivation for this definition is provided by theorem 1 in §2.

Example 1. Let $K = [0, 1]$, $w_1(x) = \frac{1}{3}x$, $w_2(x) = \frac{1}{3}x + \frac{2}{3}$, $d = 2$. Then $\{K, \mathbf{w}\}$ is an i.f.s., for any set of probabilities $\mathbf{p} = (p_1, p_2)$.
Example 2. Let $K = \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $d = 2$, and for each $z$ choose $w_1(z)$ and $w_2(z)$ to be the two solutions of

$$w^2 = z, \quad \text{for} \quad z \notin \{0, \infty\}.$$ 

Define $w_1(0) = w_2(0) = 0$, $w_1(\infty) = w_2(\infty) = \infty$. Then $\{K, w\}$ is an i.f.s. with associated probabilities $p = (\frac{1}{2}, \frac{1}{2})$.

Example 3. Let $K = \overline{\mathbb{C}}$. Let $R(w, z)$ be a two variable polynomial in $w$ and $z$, with complex coefficients and degree $d \geq 1$ in $w$ and degree $e \geq 1$ in $z$. Define $\{w_i(z): i = 1, 2, \ldots, d\}$ to be the solution of

$$R(w, z) = 0, \quad \text{for} \quad z \in \overline{\mathbb{C}},$$

where multiple roots are separated. Then $\{K, w\}$ is an i.f.s. with associated probabilities $p_i = 1/d$. We refer to this i.f.s. as the iterated Riemann surface (i.R.s.) generated by $R(w, z)$.

We can treat $w$ as a set-valued function,

$$w(x) = \{w_i(x): i = 1, 2, \ldots, d\}.$$ 

We are concerned with the action on $K$ of the set-valued iterated functions $\{w^{\circ n}(x)\}_{n=0}^{\infty}$, where

$$w^{\circ 0}(x) = x, \quad w^{\circ n}(x) = w(w^{\circ (n-1)}(x)) \quad \text{for} \quad n = 1, 2, 3, \ldots,$$

and we define, for subsets $S$ of $K$,

$$w(S) = \{w(x) : x \in S\}.$$ 

Definition 2. For given $x \in K$ we define the attractor (corresponding to $x$) for the i.f.s. $\{K, w\}$ to be

$$A(x) = \lim_{n \to \infty} w^{\circ n}(x);$$

that is, $a \in A(x)$ if and only if, whenever a neighbourhood $N$ of $a$ is specified,

$$N \cap w^{\circ n}(x) \neq \emptyset$$

for infinitely many $n$.

When $A(x)$ is independent of $x$, for values of $x$ in a distinguished subset of $K$, then we drop the reference to $x$. In such a case we refer to the attractor, $A$, of the i.f.s. $\{K, w\}$. An attractor $A$ is compact and invariant under $w$, namely $w(A) = A$.

Example 4. Let $\{K, w\}$ be the i.f.s. of example 1. Then $A$ is the classical Cantor set in $[0, 1]$, obtained by omitting middle third open intervals. $A$ is independent of $x \in K$.

Example 5. Let $\{K, w\}$ be the i.f.s. of example 2. Then for $x \in \mathbb{C} \setminus \{0\}$ we have $A(x) = \overline{\mathbb{C}} = \{z \in \mathbb{C} : |z| = 1\}$. Also $A(0) = 0$ and $A(\infty) = \infty$. $A$ is the Julia set for the iterated polynomial $f(z) = z^2$.

Example 6. Let $a$, $b$, $c \in \mathbb{C}$ be non-collinear. Consider the i.R.s. generated by (see example 3)

$$R(w, z) = (2w - z - a) (2w - z - b) (2w - z - c).$$
For all \( x \in \mathbb{C} \) the attractor \( A \) is the Sierpinski triangle with vertices at \( a, b, \) and \( c \) (figure 1).

We denote the domain and range of a function \( f \), which may be set-valued, by \( D(f) \) and \( R(f) \) respectively. For a given function \( f \) we define \( f^{-1} \) by

\[
f^{-1}(S) = \{ z \in D(f) : f(z) \in S \}.
\]

Thus, usually, \( f^{-1}(\text{elephant}) = \varnothing \).

![Figure 1. The Sierpinski lattice attractor of example 6. The picture was generated on a microcomputer in the same way as figure 2 and figures 8, 9, 10 (see text following theorem 5).](image)

**Definition 3.** Let \( \{K, w\} \) be an i.f.s such that \( \{K, w^{-1}\} \) is an i.f.s. Then \( \{K, w\} \) is an invertible i.f.s. \( B(x) \) is the repeller (corresponding to \( x \)) for \( \{K, w\} \) when, and only when, it is the attractor (corresponding to \( x \)) for \( \{K, w^{-1}\} \).

An i.R.s forms an invertible i.f.s. It is important to note that, although the individual functions \( \{w_i(z) : i = 1, 2, \ldots, d\} \) that solve \( R(w, z) = 0 \) cannot in general be defined so that they are each continuous over \( \mathbb{C} \) – it is necessary to make branch cuts to set up the domains of these functions – it is nonetheless true that \( T : C(\mathbb{C}) \to C(\mathbb{C}) \) when \( p_i = \frac{1}{d} \). In the same way the inverse functions \( \{z_j(w) : j = 1, 2, 3, \ldots, e\} \) provide an i.f.s with associated probabilities \( p_j = \hat{e} \).

**Example 7.** Let \( a, b, c \in \mathbb{C} \) be non-collinear. Consider the i.R.s. generated by

\[
R(w, z) = (w - 3z + 2a)(w - 3z + 2b)(2w - 3z + c).
\]

For any \( x \in \mathbb{C} \) the attractor is \( \infty \). For any \( x \in \mathbb{C} \) the repeller is a Cantor tree (figure 2).
Example 8. Consider the i.R.s. generated by

$$R(w, z) = w^4 - 4zw^2 + 3z^2.$$ 

We find $\mathcal{W}(z) = \{-\sqrt[3]{z}, \sqrt[3]{z}, -\sqrt[3]{3z}, \sqrt[3]{3z}\}$. The attractor (corresponding to $x \in \mathbb{C} \setminus \{0\}$) is the annulus $A = \{z \in \mathbb{C} : 1 \leq |z| \leq 3\}$. It can be described as the closure of the union of the Julia sets for all finite words obtained by composing the rational functions $f_1(z) = z^2$ and $f_2(z) = \frac{1}{3}z^2$. The repeller (corresponding to $x \in \{z \in \mathbb{C} : |z| > 3\}$) is $\infty$, and (corresponding to $x \in \{z \in \overline{\mathbb{C}} : |z| < 1\}$) is 0.

Example 9. Consider the i.R.s. generated by

$$R(w, z) = (w^8 - 9z - 81) (81w^2 - 9z - 1).$$

The attractor corresponding to $x \in \mathbb{C}$ is the probabilistic mixture of the Julia sets (Barnsley & Demko 1983) for $f_1(z) = \frac{1}{3}z^2 - 9$ and $f_2(z) = 9z^2 - \frac{9}{3}$. It looks something like the sketch in figure 3. The repeller (corresponding to some neighbourhood of $\infty$) is $\infty$.
Example 10. Let $A$ be the attractor and $B$ be the repeller (corresponding to some $x \in \mathbb{C}$) for an i.R.s. generated by $R(w, z)$. Let $M(z) = (az + \beta)/(\gamma z + \delta)$ be an invertible Möbius transformation on \overline{\mathbb{C}}. Then the i.f.s. generated by

$$(\gamma w + \delta)^d (\gamma z + \delta)^e R(M(w), M(z))$$

has the attractor $M(A)$ and repeller $M(B)$ (corresponding to $M(x)$). Because the i.R.s. generated by

$$R(w, z) = (2w - z) (2w - z - 1) (2w - z - i) (2w - z - 1 - i)$$

has the filled-in unit square with vertices 0, 1, 1+i, and i for its attractor (corresponding to $x \in \mathbb{C}$) and $\infty$ as its repeller, by using a suitable $M$ we can obtain an i.R.s. whose attractor and repeller are as sketched in figure 4.

![Figure 4. Sketch of the attractor and repeller for the i.R.s. at the end of example 10. The dotted lines lie on circles that pass through the repeller.](image)

Example 11. Consider the i.R.s. generated by

$$R(w, z) = [(w + q - 1)^b - (w - 1)^b]^{\Gamma} z^s - [(w + q - 1)^b - (w - 1)^b]^{\Gamma},$$

where $b$, $q$, $\Gamma$ and $s$ are parameters of a generalized diamond hierarchical lattice model in statistical physics (Itzykson & Luck 1983). The number of states per site is given by $q$ ($q = 2$ for an Ising model), $b$ is a constant related to how the number of bonds between sites scales under magnification and $p = \Gamma/s = b^{\Gamma - 1}$, where $d$ is the dimension. We suppose that $\Gamma$, $s$ and $b$ are integers, but permit the dimension to be fractional. The attractor for the iterated Riemann surface is the set of accumulation points of the zeros of the partition function for the model, and is relevant to the singularity structures of the associated thermodynamic functions. The associated invariant measure, discussed in subsequent sections, is of particular importance here.

Example 12. Consider the i.R.s. generated by

$$R(w, z) = \det |L + zM - wI|,$$

where $L$, $M$, and $I$ are $d \times d$ Hermitian matrices, with $I$ the identity matrix. If all of the eigenvalues of $M$ lie in $(-1, 1)$ then the attractor is a bounded subset.
of the real line and \( \{ \infty \} \) is the repeller. Example 4 is of this type, in the special case where \( L \) and \( M \) commute. In general, however, it will not be possible to write
\[
R(w, z) = R_1(w, z) R_2(w, z),
\]
where each of \( R_1 \) and \( R_2 \) is a two-variable polynomial in \( w \) and \( z \) of degree greater than zero. If such a factorization is possible, we say that the Riemann surface is reducible; otherwise it is irreducible. The present example will be irreducible if \( L \) is diagonal with distinct eigenvalues \( l_1, l_2, \ldots, l_d \), while
\[
M = \begin{bmatrix}
0 & m_1 & 0 \\
\bar{m}_1 & 0 & m_2 \\
0 & \bar{m}_2 & \ddots & \ddots \\
& & & & \bar{m}_{d-1} & 0
\end{bmatrix}
\]
with \( m_i \neq 0 \), where \( \bar{m}_i \) is the complex conjugate of \( m_i \). In this case the collection of functions \( \{ w_i(z) : i = 1, 2, \ldots, d \} \), being the eigenvalues of a non-degenerate linearly perturbed operator, may look as sketched in figure 5 for \( d = 3 \); in particular there will be no eigenvalue crossings.

![Figure 5. Sketch of a typical fan of eigenvalues, for real z, associated with the self-adjoint operator \( L + zM \), with \( d = 3 \). See example 12.](image)

Consider
\[
R(w, z) = \det \left| \begin{bmatrix}
0 & \epsilon \\
\epsilon & \frac{3}{2}
\end{bmatrix} + z \begin{bmatrix}
\frac{1}{3} & 0 \\
0 & \frac{1}{3}
\end{bmatrix} - w \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \right|,
\]
where \( \epsilon \) is real and small. Then
\[
w(z) = \{ w_1(z) = \frac{1}{3}z + O(\epsilon), \quad w_2(z) = \frac{1}{3}z + \frac{2}{3} + O(\epsilon) \},
\]
and the attractor lies very close (Hausdorff distance) to the Cantor set of
example 4, with the difference that here the Riemann surface is irreducible. One
way of viewing the genesis of the attractor in such examples is with the aid of a
bifurcating spider's web diagram, which can readily be generated on a micro-
computer and is plotted as follows. Include in the diagram in the $x,y$ plane the
curves $y = x$, $y = w_1(x)$, $y = w_2(x)$. Starting from $y_0 = x_0$, join the point $(x_0, y_0)$ to
both $(x_0, w_1(x_0))$ and $(x_0, w_2(x_0))$ with line segments parallel to the $y$ axis. Join
$(x_0, w_1(x_0))$ to $(w_1(x_0), w_1(x_0)$) with a horizontal line segment, and join $(x_0, w_2(x_0))$ to
$(w_2(x_0), w_2(x_0))$ also with a horizontal line segment. Now take both $(w_1(x_0), w_1(x_0))$
and $(w_2(x_0), w_2(x_0))$ as new starting points, and repeat the construction for each
of them. The set of $x$-coordinates $x_0$, $w_1(x_0)$, $w_2(x_0)$, $w_1(w_1(x_0))$, $w_2(w_1(x_0))$, ..., will
accumulate on the attractor (see figure 6).

![Figure 6. Bifurcating spider's web diagram approaches the attractor (x coordinates) for the
eigenvalue fan in example 12.](image)

1.3. Condensation

It turns out to be natural and interesting from the viewpoint of generating
fractals, to introduce certain modified i.f.s.s as follows. Let $\{K, w_i : i = 1, 2, \ldots, d\}$ be
an i.f.s. Let $L$ be a closed subset of $K$, and introduce the set-valued function
$w_0 : K \to \Lambda(K)$ (the subsets of $K$) by

$$w_0(x) = L \quad \text{for} \quad x \in K.$$ 

Now let $\mathbf{w} = \{w_i : i = 0, 1, \ldots, d\}$. Then we call $\{K, \mathbf{w}\} = \{K, w_i : i = 0, 1, \ldots, d\}$ an
i.f.s. with condensation. We call $L$ the condensation set.

As before, define the attractor $A(x)$ (corresponding to $x \in K$) for the i.f.s. with
condensation $\{K, \mathbf{w}\}$ to be $A(x) = \lim_{n \to \infty} \mathbf{w}^n(x)$. $A(x)$ is compact, and $\mathbf{w}(A(x)) = A(x)$. 
Example 13. Let \( K = [0, 1] \times [0, 1], \ w_1(x, y) = (\frac{1}{3}x, \frac{1}{3}y), \ w_2(x, y) = (\frac{1}{3}x + \frac{1}{2}, \frac{1}{3}y), \ w_3(x, y) = (\frac{1}{3}x + \frac{1}{4}, \frac{1}{3}y + \frac{1}{2}) \) and let \( w_0(x, y) = \) two dots and a smiling curve located inside the triangle with vertices \((\frac{1}{3}, 0), (\frac{1}{4}, \frac{1}{2})\), and \((\frac{3}{4}, \frac{1}{2})\). The attractor is sketched in figure 7.

![Figure 7. Sketch of the attractor in example 13.](image)

Definition 4. A set of probabilities

\[
P = \{ p_0, p_1, \ldots, p_d \}
\]
is said to be associated with an i.f.s. with condensation, \{\( K, w_i : i = 0, 1, \ldots, d \)\}, provided that \( 0 \leq p_0 < 1 \) and

\[
\{ \tilde{p}_i = p_i / (1 - p_0) : i = 1, 2, \ldots, d \}
\]
is a set of probabilities associated with the i.f.s. \{\( K, w_i : i = 1, 2, \ldots, d \)\}.

Here and elsewhere we reserve the function \( w_0(x) \) and the probability \( p_0 \) to be associated with condensation.

2. Balanced measures

2.1. Existence of balanced measures

Let \{\( K, w_i : i = 0, 1, \ldots, d \)\} be an i.f.s. with condensation, with associated probabilities \( P = \{ p_i : i = 0, 1, \ldots, d \} \). Let \( \sigma \) be a probability measure on \( K \), with support equal to the condensation set \( L \), so that \( \sigma(B) = \sigma(B \cap L) \) whenever \( B \in \mathcal{B}(K) \), the set of Borel subsets of \( K \), and \( \sigma(L) = 1 \).

We consider the discrete time Markov process \{\( K, w, \sigma \)\} defined by

\[
P(x, B) = \sum_{i=1}^{d} p_i \delta_{w_i(x)}(B) + p_0 \sigma(B),
\]

where \( P(x, B) \) is the probability of transfer from \( x \in K \) to the Borel subset \( B \in \mathcal{B}(K) \), and

\[
\delta_y(B) = \begin{cases} 
1 & \text{if } y \in B, \\
0 & \text{if } y \notin B.
\end{cases}
\]
We show that there is a stationary probability measure for the process; this follows the same lines as given in Barnsley & Demko (1983). The dual space of $C(K)$ is the Banach space of all finite regular signed Borel measures on $K$, which we denote by $M(K)$.

**Theorem 1.** There is a probability measure $\mu$ such that

$$\mu(B) = \int_K P(x, B) \, d\mu(x),$$

for all Borel subsets $B$ of $K$.

*Proof.* The linear operator, $T$, defined by

$$(Tf)(x) = \sum_{i=1}^{d} \frac{p_i}{1-p_0} f(w_i(x))$$

maps $C(K)$ into itself continuously. Hence the adjoint operator $T^*$ maps the set $\mathcal{P}(K)$ of probability measures on $K$ into itself weak * continuously. Hence the affine map defined on $\mathcal{P}(K)$ by

$$T^*\nu = (1-p_0) T^*\nu + p_0 \sigma$$

is a weak * continuous mapping of a weak * compact convex set into itself. By the Schauder fixed-point theorem, $T^*$ possesses a fixed point $\mu$. It is straightforward to check that

$$(T^*\nu)(B) = \sum_{i=1}^{d} \frac{p_i}{1-p_0} \nu(w_i^{-1}(B))$$

$$= \sum_{i=1}^{d} \frac{p_i}{1-p_0} \int_K \delta_{w_i(x)}(B) \, d\nu(x)$$

holds for all $B \in \mathcal{B}(K)$. It follows that

$$\mu(B) = \sum_{i=1}^{d} p_i \int_K \delta_{w_i(x)}(B) \, d\mu(x) + p_0 \sigma(B)$$

$$= \int_K P(x, B) \, d\mu(x).$$

**Definition 5.** Let $\mu$ be a stationary probability measure for the Markov process $\{K, \omega, \mathbf{p}, \sigma\}$, as in theorem 1. If $p_0 > 0$, then $\mu$ is a $\mathbf{p}$-balanced measure for the i.f.s. $\{K, \omega\}$ with condensation measure $\sigma$. If $p_0 = 0$, then $\mu$ is a $\mathbf{p}$-balanced measure for the i.f.s. $\{K, \omega\}$.

We will maintain the notation $T$, $T^*$ and $T^*$ for the operators introduced in the proof of theorem 1. Note that $T^* = T^*$ when $p_0 = 0$.

**Example 13.** Consider the i.R.S. generated by

$$Q(w) z - P(w) = R(w, z),$$

where $P(w)$ and $Q(w)$ are polynomials in $w$ with complex coefficients and $d = \max \{\deg P, \deg Q\} > 1$. $P$ and $Q$ have no common factors. The associated probabilities are $\{p_i = 1/d : i = 1, 2, \ldots, d\}$. Provided that there exists no point $c \in \mathbb{C}$ such that $w_i(c) = c$ for $i = 1, 2, \ldots, d$, that is $f(z) = P(z)/Q(z)$ has no exceptional
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points in the terminology of Brolin (1965), then $\mu$ is the $f$-invariant probability measure considered by Freire et al. (1983), who prove uniqueness. Mañé (1983) shows that $\mu$ is the unique $f$-invariant probability measure that maximizes the entropy of $f$. The support of $\mu$ is the Julia set for $f$. When $Q(z) \equiv 1$ and $f(z)$ is a polynomial, $\mu$ is the equilibrium electrostatic measure associated with $J$, see (Brolin 1965). These balanced measures have also been studied by Bessis et al. (1985), Pitcher & Kinney (1968), Barnsley et al. (1983) and Bessis & Moussa (1983).

Example 14. Consider the i.f.s. with condensation $\{K, w\}$, with $K = \overline{C}$, $d = 2$, $L = \{0\}$,

$$w_0(z) = 0, \quad w_1(z) = \sqrt{z+\lambda}, \quad w_2(z) = -\sqrt{z+\lambda},$$

where $\lambda \in \mathbb{C}$ is a parameter. The associated probabilities are $p_0 = p_1 = p_2 = \frac{1}{3}$. If the Julia set for $f(z) = z^2 - \lambda$ is hyperbolic and $\lambda \neq 0$ then the corresponding $p$-balanced measure is exactly the condensed measure considered by Barnsley et al. (1984). Its support is the Julia set for $f(z)$ together with 0 and all its images $\{ \pm \sqrt{\lambda} \pm \sqrt{\lambda} \pm ... \pm \sqrt{\lambda} \}$ under $\pm \sqrt{z+\lambda}$. The measure is distributed on its support as follows:

$$\mu(0) = \frac{1}{3}, \quad \mu(-\sqrt{\lambda}) = \mu(+\sqrt{\lambda}) = \frac{1}{3},$$

$$\mu[-\sqrt{\lambda} \pm \sqrt{\lambda}] = \mu[\pm \sqrt{\lambda} \pm \sqrt{\lambda}] = \mu[\pm \sqrt{\lambda} \pm \sqrt{\lambda}] = \frac{1}{3}, ... .$$

Note that $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + ... = 1$, showing that this accounts for all of the measure. In Barnsley et al. (1984) it is shown that if $\mu_\varepsilon$ is the balanced measure for the i.R.s. generated by

$$R(w, z) = w^3 - (\lambda + z)w + \varepsilon,$$

then $\mu_\varepsilon \rightharpoonup \mu$ as $\varepsilon \to 0$. For $\varepsilon \neq 0$ we have $\mu_\varepsilon(z) = 0$ for all $z \in \overline{C}$, in contrast to $\mu(0) = \frac{1}{3}$ for example. Hence we think of $\mu$ as being obtained from $\mu_\varepsilon$ by ‘condensation’.

Example 15. Consider the i.f.s. $\{K, w_i; i = 1, 2, 3\}$ where $K$ is a large closed disc in the complex plane, $w_1(z) = \frac{1}{2}z + \frac{1}{3}a$, $w_2(z) = \frac{1}{2}z + \frac{1}{3}b$, $w_3(z) = \frac{1}{2}z + \frac{1}{3}c$ where $a$, $b$, and $c \in K$ are not collinear. Let the associated probabilities be $p_1$, $p_2$, and $p_3$. By theorem 3 there is, in this case, a unique $p$-balanced measure. Its support is the Sierpinski triangle A with vertices a, b, and c. It is the union of three sub-triangles $w_1(A)$, $w_2(A)$ and $w_3(A)$, which have measures $p_1$, $p_2$, and $p_3$ respectively. Similarly, the subsubtriangles $w_i(w_j(A))$ have measures $p_iP_j$, and so on.

Definition 6. Let $\mu$ be a $p$-balanced measure for an i.f.s. $\{K, w\}$ and let $M \subseteq K$. Then $\mu$ is said to be attractive for measures $\nu$ on $M$ if

$$T^n \nu \rightharpoonup \mu \quad \text{as} \quad n \to \infty$$

for all probability measures $\nu$ whose support is contained in $M$.

Example 16. Let $\{\overline{C}, w_i; i = 1, 2, ..., d\}$ be the i.f.s. where the $w_i$ values are the inverse branches of a rational function $f: \overline{C} \to \overline{C}$ of degree $d > 1$, with associated probabilities $p_i = 1/d$. Let $\mu$ be the unique corresponding balanced measure whose support contains none of the exceptional points of $f$. Then it follows from Brolin (1965) when $f$ is a polynomial and from Freire et al. (1983) in the general case that $\mu$ is attractive for all measures $\nu$ whose support is contained in $\overline{C} \setminus \text{exc}(f)$, where $\text{exc}(f)$ is the set of exceptional points of $f$, of which there are at most two.
Definition 7. Let $\mu$ be a $p$-balanced measure for an i.f.s. $\{K, w\}$, and let $T$ be defined on $L_1(K, \mu)$ as in definition 1. A closed subset $G \subseteq K$ is called recurrent for $T$ if for every non-empty open (in $G)$ subset $O \subseteq G$ and every $x \in G$ there is an integer $n$ so that $(T^n I_o)(x) > 0$. Here $I_o(x) = 1$ if $x \in O$, and vanishes otherwise.

Theorem 2. Let $\mu$ be a $p$-balanced measure for an i.f.s. $\{K, w\}$ with $p > 0$. If $G \subseteq K$ is recurrent for $T$ and has $\mu(G) > 0$, then every non-empty open subset $O$ of $G$ has strictly positive measure.

Proof. Let $O \subseteq G$ be open and nonempty. Let

$$S_{k, m}^O = \left\{ z \in G : (T^m I_o)(z) > \frac{1}{k} \right\}$$

for all $k, m \in \mathbb{N}$. Then $S_{k, m}^O$ is a Borel subset of $K$, and as $G$ is recurrent we find

$$G = \bigcup_{k, m} S_{k, m}^O.$$

Because $\mu(G) > 0$, it follows that for at least one $S_{k, m}^O$ we have $\mu(S_{k, m}^O) > 0$. Hence, as $T^* \mu = \mu$,

$$\mu(O) = \int_K I_o \, d\mu = \int_K I_o \, d(T^* \mu) = \int_K T^m I_o \, d\mu > 0 \quad \text{for some } m.$$  

Example 17. Let $J$ be the Julia set associated with the i.f.s in example 16. Then it follows from Brolin (1965) that $J$ is recurrent for $T$. Hence, if we did not know it, we discover that the support of $\mu$ is $J$.

2.2. Hyperbolic iterated function systems

Some of the results here were earlier found by Hutchinson (1981).

Definition 8. $\{K, w_i : i = 1, 2, ..., d\}$ is a hyperbolic i.f.s if and only if it is an i.f.s. (with associated probabilities $p > 0$) and there is a constant $0 \leq s < 1$ such that

$$|w_i(x) - w_i(y)| < s \cdot |x - y| \quad \text{for all } x, y \in K, \quad i = 1, 2, ..., d.$$  

Here and throughout we use $|x - y|$ to denote the distance function on $K$ evaluated at $(x, y)$.

Let $\Omega$ denote the set of all half-infinite sequences of $d$ symbols, $\{1, 2, ..., d\}$, so that $\omega \in \Omega$ if and only if

$$\omega = (\omega_1, \omega_2, \ldots),$$

where $\omega_i \in \{1, 2, ..., d\}$. We reserve the notation $\omega_i$ for the $i$th component of $\omega \in \Omega$. We introduce the distance function

$$|\omega - \tilde{\omega}| = \sum_{n=1}^{\infty} \frac{|\omega_n - \tilde{\omega}_n|}{(d+1)^n} \quad \text{for } \omega, \tilde{\omega} \in \Omega.$$  

Then $\Omega$ is a compact metric space homeomorphic to the classical Cantor set.

Theorem 3. Let $\{K, w_i : i = 1, 2, ..., d\}$ be a hyperbolic i.f.s. It possesses a unique attractor $A$ (independent of $x \in K$). Moreover there exists a continuous mapping $\phi$ of $\Omega$ onto $A$, which is provided as follows. Let $\omega \in \Omega$, $n \in \mathbb{N}$, $x \in K$; define

$$w(\omega, n, x) = \omega_1 \circ \omega_2 \circ \ldots \circ \omega_n(x);$$
and then set
\[ \phi(\omega) = \lim_{n \to \infty} w(\omega, n, x), \]
where the limit exists, uniformly independent of \( x \in K \).

Proof. Recall that the attractor corresponding to \( x \in K \) is
\[ A(x) = \lim_{n \to \infty} \{ w^{\circ n}(x) \}. \]
Hence, if \( \lim_{n \to \infty} w(\omega, n, x) \) exists, it certainly belongs to \( A(x) \). That it does exist, uniformly independent of \( x \in K \) follows from
\[ |w(\omega, n, x_1) - w(\omega, m, x_2)| < s^{|m-n|} \delta(K), \quad \text{for all } x_1, x_2 \in K, \]
where \( \delta(K) = \max \{|x_1 - x_2|: x_1, x_2 \in K\} \).

Next we prove that \( \phi(\omega) \) is continuous. Let \( \epsilon > 0 \) be given. Choose \( n \) so that \( s^n \delta(K) < \epsilon \). Then whenever \( \omega, \tilde{\omega} \in \Omega \) are such that
\[ |\omega - \tilde{\omega}| < \sum_{m=n}^{\infty} \frac{d}{(d+1)^m} = \frac{1}{(d+1)^{n-1}}, \]
we have that \( \omega \) agrees with \( \tilde{\omega} \) through \( n \) terms and \( |\phi(\omega) - \phi(\tilde{\omega})| < \epsilon \).

Finally, we prove that \( \phi \) is onto. Let \( a \in A(x) \). Then there is a sequence \( \{\omega^{(n)} \in \Omega: n = 1, 2, 3, \ldots\} \) such that
\[ \lim_{n \to \infty} w(\omega^{(n)}, n, x) = a. \]

\( \{\omega^{(n)} \} \) possesses at least one limit point \( \omega \), by the compactness of \( \Omega \). Note that \( |\omega^{(n)} - \omega| \to 0 \) implies that the number of initial successive agreements between the components of \( \omega^{(n)} \) and \( \omega \) increases without limit. That is, if
\[ \alpha(n) = |\{j \in \mathbb{N}: \omega_k^{(n)} = \omega_k \quad \text{for} \quad 1 \leq k \leq j\}|, \]
then \( \alpha(n) \to \infty \) as \( n \to \infty \). Hence we find that
\[ |w(\omega, n, x) - w(\omega^{(n)}, n, x)| < s^{\alpha(n)} \delta(K). \]
It follows that \( \phi(\omega) = a \). Hence \( \phi \) is onto, and in particular \( A(x) \) is independent of \( x \in K \).

We consider properties of the special hyperbolic i.f.s. \( \{\Omega, u_i: i = 1, 2, \ldots, d\} \) where \( u_i: \Omega \to \Omega \) is defined by
\[ u_i(\omega) = (i, \omega) \quad \text{for} \quad \omega \in \Omega, \]
which means \( \omega \) is shifted to the right by one step and the symbol \( i \) placed as the first component. That is
\[ (u_i(\omega))_j = \begin{cases} i & \text{when } j = 1, \\ \omega_{j-1} & \text{when } j > 1. \end{cases} \]
In particular each \( u_i \) is continuous, with
\[ |u_i(\omega) - u_i(\tilde{\omega})| < [1/(d+1)] |\omega - \tilde{\omega}| \quad \text{for all } \omega, \tilde{\omega} \in \Omega \]
so that the system is hyperbolic. Any set of probabilities \( \mathbf{p} \) can be associated with it.
By a theorem of Daniell and Kolmogorov (see Billingsley 1965) and also theorem 4, which is to follow, there exists for each \( p > 0 \) a unique \( p \)-balanced measure \( \rho \) for the i.f.s. \( \{\Omega, u\} \). Let us use the notation \( F = \mathcal{B}(\Omega) \) for the Borel subsets of \( \Omega \). This \( \sigma \)-field can be taken to be generated by the cylinders

\[
\{\omega \in \Omega : \omega_i = i, \ n \leq l < n + k\},
\]

where each \( i \in \{1, 2, \ldots, d\} \). Then

\[
\rho(\{\omega \in \Omega : \omega_i = i, \ n \leq l < n + k\}) = \prod_{l-n}^{n+k-1} p_{il}.
\]

We will denote the analogues of the operators \( T \) and \( T^* \) for the measure space \((\Omega, F, \rho)\) by \( S \) and \( S^* \). In particular, for \( f \in C(\Omega) \), we have

\[
(Sf)(\omega) = \sum_{i=1}^{d} p_i f(u_i(\omega))
\]

and for any measure \( \nu \in M(\Omega) \) and \( B \in F \) we have

\[
(S^*\nu)(B) = \sum_{i=1}^{d} p_i \nu(u_i^{-1}(B)).
\]

(It is important to remember that for \( \omega \in \Omega \), \( u_i^{-1}(\omega) = \emptyset \) for \( \omega_i \neq i \) and \( u_i^{-1}(\omega) = \sigma \) for \( \omega_i = i \), where \( \sigma_j = \omega_{j+1} \). For a subset \( B \) of \( \Omega \) we have \( u_i^{-1}(B) = \{u_i^{-1}(\omega) : \omega \in B\} \).)

We summarize properties of \( \{\Omega, u\} \) and \( \rho \) in the following theorem.

**Theorem 4.** The i.f.s. \( \{\Omega, u_i : i = 1, 2, \ldots, d\} \) with associated probabilities \( p > 0 \), together with the probability measure \( \rho \) defined above, have the following properties:

(i) \( \{\Omega, u_i : i = 1, 2, \ldots, d\} \) is a hyperbolic i.f.s., with attractor \( A = \Omega \), independent of \( x \in \Omega \);

(ii) \( \rho \) is the unique \( p \)-balanced measure for the i.f.s.; in particular it is the unique fixed point in \( \mathcal{P}(\Omega) \) of \( S^* \), obeying \( S^*\rho = \rho \);

(iii) \( \rho \) is attractive for any probability measure \( \bar{\rho} \) on \( \Omega \), namely

\[
\lim_{n \to \infty} S^*\bar{\rho} = \rho \quad \text{for all} \quad \bar{\rho} \in \mathcal{P}(\Omega);
\]

(iv) \( \Omega \) is recurrent for \( S \); in particular the support of \( \rho \) is \( \Omega \), independently of \( p > 0 \);

(v) for all \( B \in F \),

\[
\rho(u_i(B)) = p_i \rho(B), \quad i = 1, 2, \ldots, d;
\]

(vi) for all \( g \in C(\Omega) \),

\[
\left\| S^n g - \int_{\Omega} g \, d\rho \right\|_{\infty} \to 0 \quad \text{as} \quad n \to \infty;
\]

(vii) \( S \) is mixing with respect to \( \rho \) in the sense that for all \( f, g \in C(\Omega) \)

\[
\int_{\Omega} (S^n g) f \, d\rho \to \int_{\Omega} g \, d\rho \int_{\Omega} f \, d\rho \quad \text{as} \quad n \to \infty.
\]
Proof. (i) is implied by theorem 3.
We prove (vi). We can write, for $x \in \Omega$,

$$(S^n g) (x) = \sum_{(\omega, n)} \rho(\omega, n) g(u(\omega, n, x))$$

where $u(\omega, n, x) = u_{\omega_1} \circ u_{\omega_2} \circ \ldots \circ u_{\omega_n}(x)$,

$$\rho(\omega, n) = p_{\omega_1} p_{\omega_2} \ldots p_{\omega_n} = \rho(u(\omega, n, \Omega))$$

and the sum extends over all distinct $(\omega, n) = (\omega_1, \omega_2, \ldots, \omega_n)$. Because $u(\omega, n, x) \in u(\omega, n, \Omega)$ and $\{u(\omega, n, \Omega) : (\omega, n) \text{ distinct, fixed } n\}$ is a partition of $\Omega$ whose maximum diameter tends to zero as $n \to \infty$, it follows that the right-hand side tends to $\int g \, d\rho$ uniformly in $x$. Hence (vi) is true.

(vi) implies (vii).

(iii) is proved as follows. Let $g \in C(\Omega); \quad \tilde{\rho} \in \mathcal{P}(\Omega)$; and observe, by using (vii), that

$$\int_{\Omega} g \, d(S^n \tilde{\rho} - \rho) = \int_{\Omega} S^n g \, d\tilde{\rho} - \int_{\Omega} g \, d\rho \rightarrow \int_{\Omega} \left( \int_{\Omega} g \, d\rho \right) \, d\tilde{\rho} - \int_{\Omega} g \, d\rho = 0 \quad \text{as} \quad n \to \infty.$$

(iii) implies (ii) because $\rho$ is balanced if and only if $S^\ast \rho = \rho$.

(iv) follows from the fact that $\Omega$ is the attractor for $\{\Omega, u_i : i = 1, 2, \ldots, d\}$ and the (easily proved) fact that the attractor for a hyperbolic i.f.s. is recurrent for $T$ ($T = S$ in this case).

To prove (v) recall that for all $B \in F$

$$\rho(B) = (S^\ast \rho)(B) = p_1 \rho(u_1^{-1}(B)) + p_2 \rho(u_2^{-1}(B)) + \ldots + p_d \rho(u_d^{-1}(B)),$$

so on replacing $B$ by $u_i(B)$ we obtain (v).

Now consider a hyperbolic i.f.s., $\{K, w_i : 1 \leq i \leq d\}$, with associated probabilities $\mathbf{p} > 0$ and attractor $A$. Let $T$ be the mapping of theorem 1, that is

$$(Tf)(x) = \sum_{i=1}^{d} p_i f(w_i(x)).$$

Let $\sigma \in \mathcal{P}(K)$ and consider the iterates $(T^\ast \sigma)$$_n$. The argument of theorem 3 shows that for any $\sigma \in \mathcal{P}(K)$ and any $f \in C(K)$, $\{T^n f d\sigma\}$ is a Cauchy sequence; hence $T^\ast \sigma$ converges weak * to some fixed point $\tau$ to $T^\ast$. We claim that the support of $\tau$ is contained in the attractor $A$. For if there were a closed set $B$ with $\tau(B) > 0$ and $B \cap A = \emptyset$, we could find $f \in C(K)$ with $f \equiv 1$ on $B$ and $f \equiv 0$ on an open set containing $A$ and $0 \leq f \leq 1$. Then,

$$\tau(B) \leq \int f \, d\tau = \int f \, d(T^\ast \tau) = \int T^n f \, d\tau = \sum_{(\omega, n)} \rho(\omega, n) \int f(w(\omega, n, x)) \, d\tau(x).$$

But, the argument of theorem 3 shows that $v(\omega, n, x)$ approaches $A$ uniformly in $x$ as $n \to \infty$. So, for sufficiently large $n$, $f(w(\omega, n, x)) = 0$; thus $\int f \, d\tau = 0 \leq \tau(B)$ contradicts $\tau(B) > 0$ and $B \cap A = \emptyset$. Hence, if $\tau(B) > 0$, we must have $B \subseteq A$. 
We can, therefore, restrict attention to the attractor $A$ when investigating the balanced measures for hyperbolic iterated function systems. Let $\phi: \Omega \to A$ be the continuous mapping given in theorem 3. The linear operator $\Phi: C(A) \to C(\Omega)$ defined by $(\Phi f)(\omega) = f(\phi(\omega))$ is then a one-to-one norm-one operator that satisfies $\Phi 1 = 1$. By a well-known corollary of the open mapping theorem (Berberian 1974, theorem 57.18), the adjoint operator $\Phi^*$ maps $M(\Omega)$ onto $M(A)$. In fact $\Phi^*$ maps $\mathcal{P}(\Omega)$ onto $\mathcal{P}(A)$. To see this observe that for given $\nu \in \mathcal{P}(A)$ the rule $\psi_\nu(\Phi f) = \int f \, d\nu$ defines a positive linear functional on $\Phi(C(A))$ with $\Phi^*_\nu(1) = 1$. It then has an extension to a positive linear functional $\hat{\psi}$ on $C(\Omega)$ (Berberian 1974, theorem 32.4). Because $\hat{\psi}(1) = 1$, we have $\|\hat{\psi}\| = 1$. Thus, $\hat{\psi}$ is represented by a Borel probability measure, say $\sigma$, on $\Omega$. For all $f \in C(A)$ we have $\int f \, d\nu = \int \hat{\psi} f \, d\sigma = \int \hat{\psi} d(\Phi^* \sigma)$; hence $\Phi^* \sigma = \nu$.

Let $\Phi$ be as above and $S$ as in theorem 4 and let $T$ be considered as a mapping from $C(A)$ to $C(A)$. Then one can check that $\Phi T^m = S^m \Phi$ and hence $T^{*m} \Phi^* = \Phi^* S^{*m}$ for any positive integer $m$. Now, let $\sigma \in \mathcal{P}(A)$, say $\sigma = \Phi^* \nu$, for some $\nu \in \mathcal{P}(\Omega)$. Then, $T^{*n} \sigma = T^{*n} \Phi^* \nu = \Phi^* S^{*n} \nu = \Phi^* \rho$ where $\rho$ is the unique fixed point of $S^*$ in $\mathcal{P}(\Omega)$. Hence, $\Phi^* \rho$ is the unique fixed point of $T^*$ in $\mathcal{P}(A)$ and in fact in $\mathcal{P}(K)$ by our earlier arguments. Finally, we claim $A$ is recurrent for $T$. Let $x \in K$ and $O$ be non-empty and open in $A$. Then, with $\mu = \Phi^* \rho$

$$\int T^n I_O \, d\mu = \int I_O \, d(T^{*m} \Phi^* \rho) = \int I_O \, d(\Phi^* \rho)$$

$$= \Phi \int I_O \, d\rho = \int I_{\phi^{-1}(O)} \, d\rho = \rho(\phi^{-1}(O)) > 0$$

because $\phi^{-1}(O)$ is non-empty and open in $\Omega$.

These arguments prove theorem 5.

**Theorem 5.** Let $\{K, \omega_i: i = 1, \ldots, d\}$ be a hyperbolic i.f.s. with associated probabilities $p > 0$. Then, there is a unique $p$-balanced measure, $\mu$, given by $\mu(E) = \rho(\phi^{-1}(E))$ for $E \in \mathcal{B}(K)$; $\mu$ is attractive for any probability measure $\nu$ on $K$ and the attractor $A$ is recurrent for the map $T$ (in particular, the support of $\mu$ is $A$ independently of $p > 0$).

It follows from theorem 5 that one can generate a ‘picture’ on a microcomputer video display of the $p$-balanced measure for a hyperbolic i.f.s., when $K$ is a bounded subset of $\mathbb{R}^2$ or $\mathbb{C}$, as follows. Start from any $x_0 \in K$ and choose recursively

$$x_n \in \{w_1(x_{n-1}), w_2(x_{n-1}), \ldots, w_d(x_{n-1})\}, \quad n \in \mathbb{N},$$

assigning probability $p_i$ to the choice $w_i(x_{n-1})$. Then plot successively the points

$$\{x_n: n = N, N + 1, \ldots\},$$

where $N$ is chosen sufficiently large to ensure that $x_N \in A$ to within the precision of the computation. (Alternatively, one can choose $x_0 \in A$ and take $N = 0$.) The sequence will trace out the attractor (it is delightful to watch it appear), while the density of plotted points gives an indication of the density of the measure.

The ‘pictures’ of $p$-balanced measures given in figures 8, 9 and 10, were generated in this way by using a Monroe EC8800 (240 × 240 pixels). Direct photographs from the monitor were taken in a darkened room, with Kodachrome.
Figure 8. 'Picture' of the $p$-balanced measure for a linear hyperbolic i.f.s. with $d = 3$. See the end of §2.2 and figures 9, 10.

Figure 9. 'Picture' of a $p$-balanced measure.
A.S.A. 64 colour slide film, $\frac{1}{2}$ s exposure at f/8. In each case a linear hyperbolic i.f.s. of the form

$$w_i(z) = s_i z + a_i,$$

where $s_i, a_i \in \mathbb{C}$, with $s_i \neq 1$, $|s_i| \leq 1$ and $i = 1, 2, 3$. $w_i(z)$ has fixed point

$$z_i = a_i/(1 - s_i).$$

Thus $w_i(z)$ maps circles into circles; it shrinks by $|s_i|$ and rotates by $\arg s_i$, about $z_i$. In each case the whole figure, which corresponds to part of $K$, is the square with vertices 0, 10, $10i$, $10 + 10i$; and we have $p_1 = p_2 = p_3 = \frac{1}{3}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure10.png}
\caption{‘Picture’ of a $\mathcal{P}$-balanced measure.}
\end{figure}

In figure 8,

$$s_1 = -0.5, \quad z_1 = 4 + 3i,$$
$$s_2 = -0.5, \quad z_2 = 6 + 4i,$$
$$s_3 = -0.5, \quad z_3 = 5 + 5.5i.$$  

In figure 9,

$$s_1 = 0.5i, \quad z_1 = 0,$$
$$s_2 = 0.5i, \quad z_2 = 10,$$
$$s_3 = 0.5i, \quad z_3 = 5 + 10i.$$
In figure 10,

\[ s_1 = 0.35 + 0.3i, \quad z_1 = 0, \]

\[ s_2 = 0.5 + 0.1i, \quad z_2 = 10, \]

\[ s_3 = 0.6 - 0.2i, \quad z_3 = 5 + 10i. \]

2.3. Hyperbolic i.f.s. \( \{K, w\} \) with \( w_i(A) \cap w_j(A) = \phi \) for \( i \neq j \)

It can occur that \( w_i(K) \cap w_j(K) = \phi \) for \( i \neq j \), in which case the corresponding attractor is totally disconnected and may be a Cantor set, as in example 4.

**Theorem 6.** Let \( \{K, w\} \) be a hyperbolic i.f.s. with attractor \( A, \mathcal{B} \)-balanced measure \( \mu, \mathfrak{p} > 0 \), such that \( w_i \) is one-to-one on \( A \) for \( i = 1, 2, \ldots, d \) and

\[ w_i(A) \cap w_j(A) = \phi, \quad i \neq j. \]

Then a measurable function \( Z : A \to A \) is given by

\[ Z(x) = w_i^{-1}(x) \quad \text{for} \quad x \in w_i(A). \]

It is such that \( (A, \mathcal{B}(A), \mu, Z) \) is a measure-preserving system, in the sense of Billingsley (1965), isomorphic to \( (\Omega, F, \rho, \Sigma) \) where \( \Sigma : \Omega \to \Omega \) is the Bernoulli shift operator

\[ \Sigma(\omega_1, \omega_2, \omega_3, \ldots) = (\omega_2, \omega_3, \ldots). \]

In particular \( (K, \mathcal{B}(K), \mu, Z) \) is ergodic, mixing, and has entropy

\[ h(Z) = -\sum_{i=1}^{d} p_i \ln p_i \]

**Proof.** Because \( w_i(A) \cap w_j(A) = \phi \) for \( i \neq j \), and

\[ A = w(A) = w_1(A) \cup w_2(A) \cup \ldots \cup w_d(A), \]

and as each \( w_i \) is one-to-one and continuous on \( A \), it follows that \( Z \) is a well defined measurable function taking \( A \) into itself. In particular, we find

\[ \mu(Z^{-1}(B)) = \mu \left( \bigcup_{i=1}^{d} w_i(B) \right) = \sum_{i=1}^{d} \mu(w_i(B)) \]

\[ = \sum_{i=1}^{d} p_i \mu(B) = \mu(B) \quad \text{for all} \quad B \in \mathcal{B}(A), \]

so \( \mu \) is invariant under \( Z \). Hence \( (A, \mathcal{B}(A), \mu, Z) \) is a measure-preserving system.

By using the one-to-oneness of \( w_i \) and the disjointness \( w_i(A) \cap w_j(A) = \phi \) for \( i \neq j \), it follows that the mapping \( \phi : \Omega \to A \) of theorem 3 is a homeomorphism. We readily compute much as in the proof of theorem 5,

\[ \mu(\phi(B)) = \rho(B) \quad \text{for all} \quad B \in F, \]

\[ \mu(B) = \rho(\phi^{-1}(B)) \quad \text{for all} \quad B \in \mathcal{B}(A) \]

and

\[ \phi(\Sigma \omega) = Z\phi(\omega) \quad \text{for all} \quad \omega \in \Omega. \]

Hence (Billingsley 1965), the two systems are isomorphic under \( \phi \). The last claimed properties in the statement of the theorem follow immediately because they are
invariant under isomorphism and are known for \((\Omega, F, \rho, \Sigma)\) (see Billingsley 1965).

\textbf{Example 17.} Let \(\{K, w_i: i = 1, 2\}\) be as in examples 1 and 4, with \(p_1 = p_2 = \frac{1}{2}\).

Then
\[ Z(x) = 3x \mod 1 \quad \text{for} \quad x \in A, \]
where \(A\) is the classical Cantor set. Here, \(\mu\) is the uniform measure \(A\) and \(Z\) has entropy \(\ln 2\). (Incidentally, this system is isomorphic to \(x \to 2x \mod 1\) acting on \([0, 1]\), with invariant measure equal to the uniform Lebesgue measure on \([0, 1]\) (Billingsley 1965).)

Let us suppose \(w_i(K) \cap w_j(K) = \phi\) for \(i \neq j\), under the conditions of theorem 6 and that \(w_i(K)\) is a neighbourhood of \(w_i(A)\). Then we can extend the definition of \(Z\) to
\[ N(A) = \bigcup_{i=1}^{d} w_i(K), \]
a neighbourhood of \(A\), according to
\[ Z(x) = w_i^{-1}(x) \quad \text{for} \quad x \in w_i(K), \]
so that \(Z: N(A) \to K\).

\textbf{Definition 9.} A \(k\)-cycle of \(Z: N(A) \to K\) is a set of distinct points \(\{x_1, x_2, \ldots, x_k\} \subset N(A)\) such that
\[ Z(x_1) = x_2, \quad Z(x_2) = x_3, \ldots, Z(x_k) = x_1. \]

A \(k\)-cycle \(\{x_1, x_2, \ldots, x_k\}\) of \(Z\) is repulsive when there is a set of neighbourhoods, \(N(x_i)\) of \(x_i\), with \(N(x_i) \subset N(A)\), such that
\[ Z(N(x_i)) = N(x_{i+1}), \quad i = 1, 2, \ldots, k-1, N(x_{k+1}) = N(x_1), N(x_k) \neq N(x_i). \]

\textbf{Corollary 7.} Under the above conditions, including those of theorem 6, \(A\) is the closure of the set of repulsive \(k\)-cycles of \(Z\), for all \(k \in \mathbb{N}\).

\textbf{Proof.} If \(\{x_1, x_2, \ldots, x_k\}\) is a \(k\)-cycle of \(Z\), then it is contained in \(A\), by the uniqueness of \(A\) (theorem 3). Because the i.f.s. is hyperbolic it follows that any \(k\)-cycle of \(Z\) is repulsive. \(\Omega\) is the closure of the set of all \(k\)-cycles for \(\Sigma: \Omega \to \Omega\), and the homeomorphism \(\phi: \Omega \to A\) is such that \(\phi \circ \Sigma = Z \circ \phi\), it follows that \(A\) is the closure of the set of all \(k\)-cycles for \(Z\).

\textbf{Example 18.} Let \(\lambda > 0\) be such that
\[ 0 < \lambda - \frac{3}{4} - \sqrt{(\lambda + 1/2)}. \]
Let \(K = [-a, -b] \cup [b, a]\), where
\[ a = \frac{1}{4} + \sqrt{(\lambda + 1/4)}, \quad b = \sqrt{(\lambda - a)}. \]

Let \(d = 2\) and choose
\[ w_1(x) = \sqrt{(\lambda + x)}, \quad w_2(x) = -\sqrt{(\lambda + x)}. \]
Then \(\{K, w_i: i = 1, 2\}\) is a hyperbolic i.f.s. which obeys the conditions of corollary 7. Hence \(A\) is the closure of the set of repulsive \(k\)-cycles of \(Z\). In this case \(A\) is the Julia set for \(f(z) = z^2 - \lambda\), and the action of \(Z\) on \(N(A)\) is the same as that of \(f(z)\).
Next we consider the Hausdorff–Besicovitch dimension of subsets $A$ of $K$. For $0 \leq p < \infty$, let

$$M_p^e(A) = \inf \sum_{i=1}^{\infty} [\delta(A_i)]^p,$$

where $A = \bigcup_{i=1}^{\infty} A_i$ is a countable decomposition of $A$ into subsets of diameter $\delta(A_i)$ less than $\varepsilon > 0$. Here $\delta(A_i) = \sup\{|x-y| : x, y \in A_i\}$. We set $[\delta(A_i)]^0 = 0$ if $A_i$ is empty and $[\delta(A_i)]^0 = 1$ otherwise. Then the $p$-dimensional measure of $A$ is defined to be

$$M_p(A) = \sup_{\varepsilon > 0} M_p^e(A).$$

The Hausdorff–Besicovitch dimension of $A$ (see Hurewitz & Wallman 1948) is

$$\hat{p}(A) = \sup\{0 \leq p < \infty : M_p(A) > 0\}.$$

The following theorem is a pleasing application of theorem 1.

**Theorem 8.** Let $\{K, w_i : i = 1, 2, \ldots, d\}$ be a hyperbolic i.f.s. with attractor $A$, with $w_i(A) \cap w_j(A) = \emptyset$ for $i \neq j$, and with $K$ a subset of $\mathbb{R}^n$, the metric being the usual one on $\mathbb{R}^n$. Assume there are numbers $0 < s_i \leq \hat{s}_i < 1$ such that

$$s_i |x-y| \leq |w_i(x) - w_i(y)| \leq \hat{s}_i |x-y|, \quad i = 1, 2, \ldots, d.$$

Then the Hausdorff–Besicovitch dimension $\hat{p}(A)$ of $A$ is bounded by

$$\min\{n, l\} \leq \hat{p}(A) \leq u,$$

where $l$ and $u$ are the positive solutions of

$$\sum_{i=1}^{d} s_i^l = 1 \quad \text{and} \quad \sum_{i=1}^{d} \hat{s}_i^u = 1.$$

If $w_i(A) \cap w_j(A) \neq \emptyset$ for some $i \neq j$, the upper bound remains valid.

**Proof.** To get the upper bound, let $A = \bigcup_{i=1}^{\infty} A_i$ be any decomposition of $A$ into subsets of diameter $< \varepsilon$. Then a new decomposition is provided by

$$A = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{d} A_{ij},$$

where $A_{ij} = w_j(A_i)$. Because

$$\sum_{i=1}^{\infty} \sum_{j=1}^{d} [\delta(A_{ij})]^p \leq \sum_{i=1}^{\infty} \sum_{j=1}^{d} |s_j|^p [\delta(A_i)]^p$$

$$= \left(\sum_{j=1}^{d} |s_j|^p\right) \sum_{i=1}^{\infty} [\delta(A_i)]^p,$$

it follows that whenever $\sum_{j=1}^{d} |s_j|^p < 1$ we must have $M_p^e(A) = 0$, which implies $M_p(A) = 0$. As $\hat{p}(A) = \inf\{p : M_p(A) = 0\}$ (see Blumenthal & Getoor 1960), the upper bound follows at once.

To get the lower bound we use a theorem attributed by Blumenthal & Getoor (1960) to H. P. McKean. In the present context it states that if $\mu$ is any probability measure on $A$ and if $\beta$ is such that

$$\int \int \frac{d\mu(x) d\mu(y)}{|x-y|^\beta} < \infty,$$

then $\beta \leq \hat{p}(A)$. 
Without loss of generality we can suppose that $K$ is a ball in $\mathbb{R}^n$, because we can always extend the domains of $w_i$ to a ball in such a way as to maintain a hyperbolic i.f.s. with $w_i(K) \cap w_j(K) = \emptyset$ for $i \neq j$. Let $\mu_0$ be the uniform probability measure on $K$, so that

$$I_0 = \int_K \int_K |x - y|^{-\beta} \, d\mu_0(x) \, d\mu_0(y) < \infty$$

for $0 \leq \beta < n$. Then define

$$I_{n+1} = \int_K \int_K |x - y|^{-\beta} \, d\mu_{n+1} \, d\mu_{n+1}, \quad n = 0, 1, 2, \ldots,$$

where

$$\mu_{n+1}(B) = \sum_{i=1}^d p_i \mu_n(w_i^{-1}(B)) = (T^* \mu_n)(B)$$

for all $B \in \mathcal{B}(K)$. Here $\mathbf{p} = (p_1, p_2, \ldots, p_d)$ is an arbitrary set of probabilities; any such set can be associated with the system because the $w_i(x)$ values are continuous. By theorem 5, $\mu_n \to \mu$, the corresponding $\mathbf{p}$-balanced measure, as $n \to \infty$.

Now

$$I_{n+1} = \int_K \int_K |x - y|^{-\beta} \, d\mu_{n+1}(x) \, d\mu_{n+1}(y)$$

$$= \sum_{i=1}^d \sum_{j=1}^d p_i p_j \int_K \int_K |w_i(x) - w_j(y)|^{-\beta} \, d\mu_n(x) \, d\mu_n(y)$$

$$= \sum_{i=1}^d p_i^2 \int_K \int_K |w_i(x) - w_i(y)|^{-\beta} \, d\mu_n(x) \, d\mu_n(y)$$

$$+ \sum_{i, j=1 \atop i \neq j}^d p_i p_j \int_K \int_K |w_i(x) - w_j(y)|^{-\beta} \, d\mu_n(x) \, d\mu_n(y).$$

The first term here is dominated by

$$\sum_{i=1}^d p_i^2 s_i^{-\beta} I_n = C(\beta) I_n$$

and the second term is dominated by

$$\sum_{i, j=1 \atop i \neq j}^d p_i p_j \rho^{-\beta} = d(\beta) < \infty,$$

where

$$\rho = \min \{|x - y| : x \in w_i(K), y \in w_j(K), i \neq j\}.$$

Hence for $0 \leq \beta < n$,

$$I_{n+1} \leq C(\beta) I_n + d(\beta)$$

$$\leq (C(\beta))^{n+1} I_0 + \sum_{j=0}^n (C(\beta))^j d(\beta).$$
Global construction of fractals

It follows that if $0 < C(\beta) < 1$ then $\{I_n: n = 0, 1, 2, \ldots\}$ is uniformly bounded; whence

$$\int_A \int_A |x-y|^{-\beta} \, d\mu(x) \, d\mu(y) < \infty.$$ 

Hence

$$\hat{p}(A) \geq \sup \left\{ \beta < n: \sum_{i=1}^{d} p_i^\beta s_i^{-\beta} < 1 \right\},$$

from which we obtain

$$\hat{p}(A) \geq \min\{n, \beta\},$$

where $\beta$ is the positive solution of

$$\sum_{i=1}^{d} p_i^\beta s_i^{-\beta} = 1.$$ 

We now maximize $\beta$ over all possible $p$, by using the method of Lagrange multipliers and yielding

$$\hat{p}(A) \geq \min\{n, l\},$$

where $l$ is the positive solution of

$$\sum_{i=1}^{d} s_i^l = 1.$$ 

It is interesting to note that the choice of probabilities that yields the lower bound at the end of this proof is

$$p_i = \frac{s_i^l}{\left(\sum_{i=1}^{d} s_i^l\right)} = s_i^l, \quad i = 1, 2, \ldots, d.$$ 

Example 19. Consider a linear hyperbolic i.f.s. acting on a subset $K$ of $\mathbb{C}$ (which is equivalent to $\mathbb{R}^2$) according to

$$w_i(z) = s_i z + a_i, \quad i = 1, 2, \ldots, d,$$

where the $s_i$ and $a_i$ values are complex numbers with $0 < |s_i| < 1$ for each $i$. Assume that $w_i(K) \cap w_j(K) = \phi$ for $i \neq j$. Then in theorem 8 we can take $s_i = \bar{s}_i = |s_i|$, and it is easy to see that the upper bound $u$ must be less than or equal to 2. Hence $u = l$, whence the Hausdorff dimension is exactly the positive solution of

$$\sum_{i=1}^{d} |s_i|^{\hat{p}(A)} = 1.$$ 

(Conversely, for this linear hyperbolic i.f.s., if we found on solving the latter equation that $\hat{p}(A) > 2$, we would have a contradiction, implying that $w_i(A) \cap w_j(A) \neq \phi$ for some $i \neq j$.) If we take $d = 2, s_1 = s_2 = \frac{1}{2}, a_1 = 0, \text{ and } a_2 = \frac{2}{3}$; then $A$ is the classical Cantor set, with dimension $\hat{p}$ which obeys

$$\frac{2}{3^p} = 1;$$

that is, $\hat{p} = \ln 2 / \ln 3$, as is well known (Hurewitz & Wallman 1948).
3. Moment Theory

3.1. Moment Theory of linear i.f.s. in $C^n$ and $R^n$

We make the discussion in $C^n$ for brevity; the results are readily transcribed to $R^n$. Let $K$ be a compact subset of $C^n$, with the usual metric, and let $L$ be a closed subset of $K$. Then we consider a special class of i.f.s. \{$K, w_i : i = 0, 1, \ldots, d$\} with condensation. $L$ is the condensation set. Let affine maps \(w_i : K \to K, i = 1, 2, \ldots, d\), be defined by

\[w_i z = S_i z + (1 - S_i) a_i \quad \text{for} \quad i = 1, 2, \ldots, d,\]

where \(z = (z_1, z_2, \ldots, z_n)^t\) is a column vector in $K$, $t$ denotes the transpose and

\[S_i = \begin{bmatrix}
s_{i1} & 0 & 0 \\
0 & s_{i2} & 0 \\
0 & 0 & \ddots \\
0 & \ddots & \ddots & s_{in}
\end{bmatrix}\]

is a diagonal matrix with $s_{ij} \in C$ and $|s_{ij}| < 1$, and the $a_i$ values are points in $C^n$,

\[a_i = (a_{i1}, a_{i2}, \ldots, a_{in})^t.\]

Note that \(w_i a_i = a_i\) so that $w_i$ is specified by its magnification matrix, $S_i$, and its fixed point $a_i$.

Let $\sigma$ be any probability measure whose support is $L$, and let

\[\mathcal{P} = \{p_0, p_1, \ldots, p_d\}\]

be a set of probabilities with $0 \leq p_0 < 1$ and $p_i > 0$ for $i = 1, 2, \ldots, d$. Let $\mu$ be a $\mathcal{P}$-balanced measure for the i.f.s. \{$K, w$\} with condensation measure $\sigma$. We recall that for all $f \in L^1(K, \mu)$, we have

\[\int_K f d\mu = \sum_{i=1}^d p_i \int_K f \circ w_i d\mu + p_0 \int_K f d\sigma. \quad (\dagger)\]

We introduce the moments of $\mu$. We use the notation

\[z^m = z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n},\]

where \(m = (m_1, m_2, \ldots, m_n) \in \mathbb{N}_0^n, \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\}\).

We denote the moments of $\nu \in \mathcal{P}(K)$ by

\[M(\nu, m) = \int_K z^m d\nu(z), \quad \text{for all} \quad m \in \mathbb{N}_0^n.\]

**Theorem 9.** Let $\mu$ be a $\mathcal{P}$-balanced measure for the i.f.s. \{$K, w_i : i = 0, 1, \ldots, d$\} with condensation measure $\sigma$, defined above. Let

\[\max \{\min \{|s_{ij}| : i = 1, 2, \ldots, d\} : j = 1, 2, \ldots, n\} < 1.\]

Then the moments

\[\{M(\mu, m) : m \in \mathbb{N}_0^n\}\]
Global construction of fractals can be calculated uniquely explicitly recursively, in terms of the moments \( \{ M(\sigma, m) : m \in \mathbb{N}_0^n \} \) and the parameters which define the i.f.s. The manner in which this can be done is given in the proof which follows.

**Proof.** Because \( K \) is bounded it follows that \( z^m \in L_1(K, \mu) \). Hence, on choosing \( f = z^m \) in (1) above we find that for \( m \neq 0 \),

\[
M(\mu, m) = \int_K z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d} \, d\mu(z)
\]

\[= \sum_{i=1}^d p_i \int_K (s_{i1} z_1 + a_{i1})^{m_1} (s_{i2} z_2 + a_{i2})^{m_2} \cdots (s_{in} z_n + a_{in})^{m_n} \, d\mu + p_0 M(\sigma, m).\]

We can rewrite this in the form

\[
\left(1 - \sum_{i=1}^d p_i \prod_{j=1}^n s_{ij}^{m_j}\right) M(\mu, m) = \sum_{0 \leq k \leq m, k \neq m} C(k) M(\mu, k) + p_0 M(\sigma, m),
\]

where the sum extends over all \( k = (k_1, k_2, \ldots, k_n) \) with \( 0 \leq k_j \leq m_j, j = 1, 2, \ldots, n \), with \( k \neq m \), and where each \( C(k) \) is a complex number which can be written explicitly in terms of \( p_0, a_{ij}, s_{ij} \) and \( m_j \). The conditions in the theorem ensure that the coefficient of \( M(\mu, m) \) on the left does not vanish, whence we can solve uniquely, explicitly for \( M(\mu, m) \) in terms of

\[\{M(\mu, k) : 0 \leq k \leq m, k \neq m\}.\]

As \( M(\mu, 0) = \int_K d\mu = 1 \), \( M(\mu, m) \) can be calculated recursively.

**Corollary 10.** Under the conditions of theorem 9, in the real case where \( K \subset \mathbb{R}^n \), \( a_{ij} \in \mathbb{R} \) and \( s_{ij} \in \mathbb{R} \) for \( i = 1, 2, \ldots, d \), and \( j = 1, 2, \ldots, n \), the associated moment problem is determinate. In particular, the corresponding \( p \)-balanced measure \( \mu \) is unique.

**Proof.** Because the support of the probability measure \( \mu \) is bounded and real, the moment problem is determinate; that is the problem of finding \( \mu \) given its moments \( \{ M(m) : m \in \mathbb{N}_0^n \} \) has a unique solution (Shohat & Tamarkin 1950; Akhiezer 1965). By theorem 9, the \( M(m) \) values are unique.

**Example 20.** Consider the i.f.s. as in example 15,

\[w_1(z) = \frac{1}{2} z, \quad w_2(z) = \frac{1}{2} z + \frac{1}{2}, \quad w_3(z) = \frac{1}{2} z + \frac{1}{2} i,\]

which has the Sierpinski gasket with vertices at 0, 1, and \( i \), as its attractor. If we view the system as acting in \( \mathbb{C} \) (\( n = 1 \)) then theorem 9 says we can calculate the moments

\[M(\mu, m) = \int_K z^m \, d\mu, \quad m = 0, 1, 2, \ldots\]

for any \( p \)-balanced measure \( \mu \) for the i.f.s., without or with a condensing measure \( \sigma \).

However, in this example, we can also treat each \( w_i \) as a map from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \), according to

\[w_1(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} \frac{1}{2} x \\ \frac{1}{2} y \end{pmatrix}, \quad w_2(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} \frac{1}{2} x + \frac{1}{2} \\ \frac{1}{2} y \end{pmatrix}, \quad w_3(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} \frac{1}{2} x \\ \frac{1}{2} y + \frac{1}{2} \end{pmatrix} \]
Theorem 9 now tells us we can actually calculate

\[ M(m_1, m_2) = M(\mu, m_1, m_2) = \int_K x^{m_1} y^{m_2} d\mu(x, y) \quad \text{for all} \quad m_1, m_2 \in \mathbb{N}_0. \]

In fact, for \( p_0 = 0, p_1 = p_2 = p_3 = \frac{1}{3} \) we find

\[
M(m_1, m_2) = 3^{-(m_1+m_2+1)} \left( M(m_1, m_2) + \sum_{j=0}^{m_1} M(m_1-j, m_2) \binom{m}{j} 2^j \right.
\]
\[
+ \sum_{k=0}^{m_2} M(m_1, m_2-k) \binom{m_2}{k} 2^k \right). \]

In such cases we can actually calculate all of the orthogonal polynomials \( \{P_n(z) : n = 0, 1, 2, \ldots\} \) which obey

\[
\int P_n(z) P_m(z) d\mu(z) = 0, \quad m \neq n,
\]

where \( P_n(z) \) has degree \( n \) and unit leading coefficient. This contrasts with the situation associated with Julia sets for polynomials (see Barnsley et al. 1982).

Let \( \mu \) be a \( p \)-balanced measure for the i.f.s. \( \{K, \omega\} \) with condensation measure \( \sigma \), with \( K \) a bounded subset of \( \mathbb{C} \) and

\[ w_i(z) = s_i z + a_i \quad i = 1, 2, \ldots, d, \]

with \( s_i, a_i \in \mathbb{C}, s_i \neq 0 \). Let us introduce the Stieltjes transforms

\[
f_\beta(z) = \int \frac{d\mu(\eta)}{(z-\eta)^\beta}; \quad g_\beta(z) = \int \frac{d\sigma(\eta)}{(z-\eta)^\beta},
\]

which are defined for \( z \in \mathbb{D} = \mathbb{C} \setminus \{\text{support of } \mu\} \). Then, exploiting the \( p \)-balanced property of \( \mu \), we find that \( f_\beta(z) \) obeys the functional equation

\[
f_\beta(z) = \sum_{i=1}^{d} \frac{p_i}{s_i^\beta} f_\beta(w_i^{-1}(z)) + p_0 g_\beta(z), \quad z \in \mathbb{D}.
\]

Conversely, the existence of \( p \)-balanced measures (theorem 1) shows the existence of solutions to these functional equations, which are of a form of interest in statistical physics (Derrida et al. 1983).

**Example 21.** Let us take \( K = [0, 1] \), with

\[ w_1(z) = \frac{1}{2} z, \quad w_2(z) = \frac{1}{2} z + \frac{3}{4}, \quad p_1 = p_2 = \frac{1}{2}. \]

Then, associated with the uniform measure on the Cantor set we have the Stieltjes transform \( f_\beta(z) \), which obeys

\[
f_\beta(z) = \frac{3^\beta}{2} (f_\beta(3z-2) + f_\beta(3z)), \quad \text{for all} \quad z \in \mathbb{C} \setminus A.
\]

We can readily estimate the behaviour of \( f(z) \) near its singularities. For example, choose \( \beta = 1 \) and make the ansatz that, near \( z = 1 \),

\[ f(z) = C \cdot (z-1)^{-\delta} + \text{terms } O(z-1)^{-\delta+1}. \]
Then we find

\[ C \cdot (z-1)^{-\delta} = \frac{3}{2} \cdot C \cdot 3^{-\delta} (z-1)^{-\delta} + f(3z) + \text{terms } O(z-1)^{-\delta+1}, \]

from which

\[ \delta = \ln \left( \frac{3}{2} \right) / \ln 3. \]

### 3.2. Moment theory of i.R.s.

We begin by considering the moment theory of iterated Riemann surfaces (see example 3), with condensation. Let \( R(w, z) \) be a two-variable polynomial in \( w \) and \( z \), with complex coefficients, of the special form

\[ R(w, z) = \sum_{n=0}^{d} P_n(z) w^{d-n}, \]

where \( P_n(z) \) is a polynomial in \( z \) of degree at most \( n \) and \( P_0(z) = 1 \). Let

\[ q_n = \text{coefficient of } z^n \text{ in } P_n(z), \quad n = 0, 1, 2, \ldots, d. \]

Let \( \{w_i(z) : i = 1, 2, \ldots, d\} \) be a complete assignment of solutions of

\[ R(w, z) = 0, \]

with multiplicities included, so that each \( w_i(z) \) is a mapping

\[ w_i : \overline{C} \to \overline{C} \quad \text{where} \quad \overline{C} = C \cup \{\infty\}. \]

Let \( K = \overline{C} \), and \( \sigma \) be any probability measure on \( \overline{C} \). Let the support of \( \sigma \) be \( L \). Then we consider the i.R.s. \( \{\overline{C}, w_i : i = 0, 1, \ldots, d\} \) with condensing measure \( \sigma \) and associated probabilities \( 0 \leq p_0 < 1 \) and \( p_i = (1-p_0)/d \) for \( i = 1, 2, \ldots, d \). Let \( \mu \) denote a corresponding \( p \)-balanced measure. Then we study the moments

\[ M(\mu, n) = \int_{\overline{C}} z^n \, d\mu(z), \quad n \in \mathbb{N}_0. \]

**Theorem 11.** As above, let \( \mu \) be a \( p \)-balanced measure for the i.R.s. \( \{\overline{C}, w_i : i = 0, 1, 2, \ldots, d\} \), with condensation measure \( \sigma \). Let \( \{e_i : i = 1, 2, \ldots, d\} \) denote the roots of

\[ \sum_{n=0}^{d} q_n w^{d-n} = 0, \]

repeated according to their multiplicities. Let \( m \in \mathbb{N}, \)

\[ \frac{(1-p_0)}{d} \sum_{i=1}^{d} e_i^j \neq 1 \quad \text{for} \quad j = 1, 2, \ldots, m, \]

\( z^m \in L_1(\overline{C}, \mu) \) and \( z^m \in L_1(\overline{C}, \sigma) \). Then \( M(\mu, m) \) can be calculated uniquely, explicitly, recursively in terms of

\[ \{M(\sigma, n) : n = 0, 1, 2, \ldots, m\} \]

and the parameters that define the i.R.s. The manner in which this can be done is given in the proof which follows.
Proof. Because \( z^n \) belongs to both \( L_1(\mathbb{C}, \sigma) \) and \( L_1(\mathbb{C}, \mu) \) it follows that the moments \( M(\sigma, n) \) and \( M(\mu, n) \) exist for all \( n = 1, 2, 3, \ldots, m \) and we can choose \( f = z^n \) in the equation (which expresses the \( \varphi \)-balanced property of \( \mu \)),
\[
\int f \, d\mu = \frac{(1-p_0)}{d} \int \sum_{l=1}^{d} f(w_l) \, d\mu + p_0 \int f \, d\sigma,
\]
which yields
\[
M(\mu, n) = \frac{(1-p_0)}{d} \int S_n(z) \, d\mu(z) + p_0 \, M(\sigma, n), \tag{*}
\]
where \( S_n(z) \) is the symmetric function
\[
S_n(z) = \sum_{l=1}^{d} (w_l(z))^n
\]
which is associated with the roots \( \{w_l : l = 1, 2, \ldots, d\} \) of the equation
\[
R(w, z) = 0.
\]
These symmetric functions obey the recursion relations (see Gaal 1971)
\[
S_n(z) = -nP_n(z) - \sum_{k=1}^{n-1} S_k(z) \, P_{n-k}(z), \quad n = 1, 2, \ldots, M, \ldots,
\]
from which it follows by induction that \( S_n(z) \) is a polynomial in \( z \) of degree at most \( n \). Moreover, the coefficient of \( z^n \) in \( S_n(z) \) is easily seen to be
\[
\Gamma_{nn} = \sum_{l=1}^{d} \epsilon_l n,
\]
where we write
\[
S_n(z) = \Gamma_{nn} z^n + \Gamma_{n, n-1} z^{n-1} + \ldots + \Gamma_{n, 0}.
\]
Hence we can rewrite \((*)\) as
\[
\left(1 - \Gamma_{n, n} \frac{(1-p_0)}{d}\right) M(\mu, n) = \frac{(1-p_0)}{d} \sum_{j=0}^{n-1} \Gamma_{n, j} M(\mu, j) + p_0 \, M(\sigma, n).
\]
Because the coefficient of \( M(\mu, n) \) on the left-hand side does not vanish, and we know \( M(\mu, 0) = 1 \); we can solve for \( M(\mu, 1), M(\mu, 2), \ldots, M(\mu, m) \), exactly as claimed in the statement of the theorem. \( \square \)

Example 22. When deg \( P_n(z) \leq 1 \) for each \( n = 0, 1, \ldots, m \) and \( p_0 = 0 \), the situation cited in the theorem becomes exactly the one considered by Barnsley & Harrington (1984). In this case, \( \mu \) can be the balanced measure on the Julia set for the rational function, assumed to be of degree \( d \geq 2 \),
\[
f(z) = - \sum_{n=0}^{d} \frac{P_n(0)}{d} \frac{z^n}{P'_n(0) \, z^n}.
\]
In Barnsley & Harrington (1984) it is proved that the conditions of the present theorem are met when \( \infty \) is an attractive or indifferent fixed point of \( f(z) \). In fact, when \( P'_d(0) = 0 \), \( z^n \in L_1(\mathbb{C}, \mu) \) if and only if \( \infty \) is an attractive or indifferent fixed point of \( f(z) \).
Corollary 12. There is at most one $\mathbf{p}$-balanced measure $\mu$ for the i.R.s. 
$\{\mathbb{C}, w_i: i = 0, 1, 2, \ldots, d\}$ with condensation measure $\sigma$, such that the conditions of 
theorem 11 are met, and the support of $\mu$ is a bounded subset of the real line.

Proof. This is essentially the same as that of corollary 10.

Example 23. Consider the i.R.s. generated by

$$
R(w, z) = \det \begin{vmatrix} z - w & 3\varepsilon \\
3\varepsilon & 2 + z - 3w \end{vmatrix} = 0
$$

where $\varepsilon > 0$ is a small number, as in example 12. One readily verifies that

$$
|\text{Im } w_i(z)| < |\text{Im } z| \quad \text{when } \text{Im } z \neq 0,
$$

whence, whenever the condensation measure $\sigma$ has real support $L \in \mathbb{R}$, we find that 
the support of any $\mathbf{p}$-balanced measure $\mu$ must be real and bounded. Furthermore,

$$
P_0(z) = 1, \quad P_1(z) = -\frac{3}{8}(1 + z), \quad P_2(z) = z(2 + z),
$$

whence the roots of

$$
q_0 w^2 + q_1 w + q_2 = w^2 - \frac{3}{2}w + \varepsilon^2 = 0
$$

are $\varepsilon_1 = \text{order } \varepsilon$, and $\varepsilon_2 = \frac{3}{2} + \text{order } \varepsilon$. Hence

$$
\frac{1}{2}(1 - p_0) (\varepsilon_1^j + \varepsilon_2^j) < 1 \quad \text{for } j = 1, 2, \ldots;
$$

the conditions of theorem 11 and corollary 12 are met and we conclude that $\mu$ is 
the unique $\mathbf{p}$-balanced measure with condensation such that the support of $\mu$ is 
contained in $\mathbb{C}$.

Finally, we generalize theorem 11 to cover the case of probabilistic mixtures of 
i.R.s. This extends the idea of probabilistic mixtures of Julia sets introduced in 
Barnsley & Demko (1983). We consider an i.f.s. of the special form

$$
\{\mathbb{C}, w_0, w_{ij}: j = 1, 2, \ldots, d_i, \quad i = 1, 2, \ldots, h\},
$$

where for each $i$ the functions $\{w_{ij}: j = 1, 2, \ldots, d_i\}$ are a complete assignment of 
solutions of

$$
R_i(w, z) = 0.
$$

Here $R_i(w, z)$ is assumed to generate an i.R.s., and to be of the special form

$$
R_i(w, z) = \sum_{j=0}^{d_i} p_{ij}(z) w^{d_i-j},
$$

where $P_{ij}(z)$ is a polynomial in $z$ of degree at most $j$, and $P_{ij}(z) = 1$. Let $\{\varepsilon_{ij}: 
\varepsilon_{ij}: j = 1, 2, \ldots, d_i\}$ denote the roots of the polynomial in $w$,

$$
\sum_{j=0}^{d_i} q_{ij} w^{d_i-j} = 0.
$$

We introduce the condensation measure $\sigma$ with support $L \subset \mathbb{C}$, and probabilities

$$
0 \leq p_0 < 1 \quad \text{and } p_{ij} = p_i/d_i \quad \text{for } j = 1, 2, \ldots, d_i;
$$

where

$$
\sum_{ij} p_{ij} = \sum_{i=1}^{h} p_i = (1 - p_0).
$$

The proof of the following theorem is a simple extension of that of theorem 11.
THEOREM 13. As above, let \( \mu \) be a \( p \)-balanced measure for the probabilistic mixture of i.R.s. \( \{ \overline{C}, w_o, w_i, j = 1, 2, \ldots, d_i, i = 1, 2, \ldots, h \} \) with condensation measure \( \sigma \). Let \( m \in \mathbb{N} \),

\[
(1-p_0) \sum_{i-1}^{h} \frac{P_i}{d_i} \sum d_i \neq 1 \quad \text{for} \quad j = 1, 2, \ldots, m,
\]

\( z^m \in L_1(\overline{C}, \mu) \) and \( z^m \in L_1(\overline{C}, \sigma) \). Then \( M(\mu, m) \) can be calculated uniquely, explicitly, recursively in terms of

\[
\{M(\mu, n) : n = 0, 1, 2, \ldots, m\}
\]

and the parameters that define the system.

In Barnsley & Demko (1983) we consider the case of probabilistic mixtures of Julia sets. Conditions are given, such that the support of a \( p \)-balanced measure may be \( \mathbb{R} \cup \{\infty\} \), yet all of the moments \( \{M(\mu, m)\}_{m=0}^{\infty} \) exist and correspond to a determinate moment problem (they must not grow too fast). In such cases the uniqueness of the measure can be inferred. It is quite clear that such results can be extended to probabilistic mixtures of i.R.s.

In a variety of cases it is possible to obtain functional equations obeyed by Stieltjes transforms of \( p \)-balanced measures, for probabilistic mixtures of i.R.s. with condensation. A class of such examples, involving the Julia set for a rational function \( f(z) \) and a condensation set consisting of a finite sets of points, was introduced in Barnsley et al. (1983) and shown to have an application in describing the spectrum of a tight-binding hamiltonian on a Sierpinski lattice. Here we simply illustrate the more general type of functional equation, solutions of which can be exhibited in terms of \( p \)-balanced measured associated with probabilistic mixtures.

Example 24. Consider the probabilistic mixture of i.R.s., with condensation measure \( \sigma \), associated with

\[
R_1(w, z) = w^2 - (2+z), \quad R_2(w, z) = w - \frac{1}{z}.
\]

We assume that the condensation set \( L \) obeys \( L \leq [-2, 2] \). Then it is readily seen by choosing \( K = [-2, 2] \) that the system admits a \( p \)-balanced measure \( \mu \) supported in \([ -2, 2 ] \). Then for all \( f \in L_1([-2, 2], \mu) \) we have

\[
\int_{-2}^{2} f(\eta) d\mu(\eta) = \frac{P_1}{2} \int_{-2}^{2} f(\sqrt{(2+\eta)}) d\mu(\eta) + \frac{P_1}{2} \int_{-2}^{2} f(-\sqrt{(2+\eta)}) d\mu(\eta)
\]

\[
+ p_2 \int_{-2}^{2} f(\frac{1}{\eta}) d\mu(\eta) + p_0 \int_{-2}^{2} f(\eta) d\sigma(\eta).
\]

Let \( f(\eta) = (z-\eta)^{-1} \) where \( \eta \in [-2, 2] \) and \( z \in \mathbb{C} \setminus [-2, 2] \). Then the Stieltjes transform

\[
F(z) = \int_{-2}^{2} \frac{d\mu(\eta)}{(z-\eta)}
\]

obeys the functional equation

\[
F(z) = p_1 \cdot z \cdot F(z^2 - 2) + 2p_2 F(2z) + p_0 H(z),
\]

where

\[
H(z) = \int_{-2}^{2} \frac{d\sigma(\eta)}{(z-\eta)}.
\]
3.3. Fractal reconstruction

We give an example to demonstrate the feasibility of reconstructing, approximately or exactly, given fractal structures with the aid of linear i.f.s. and moment theory. We consider the twin-dragon fractal (Davis & Knuth 1970; Mandelbrot 1982). Figure 11 illustrates this fractal, which has dimension 2. This figure was not generated by the method of previous workers, involving successive refinements – eventually microscopic – of a curve made of straight line segments; instead it was obtained by using an i.f.s. (see Barnsley & Harrington 1985 and this paper). We describe how we are able to approximate this fractal \( F \), with the aid of partial information about its moments and symmetry.

![Figure 11. The twindragon fractal \( F \). See §3.3.](image)

We commence with the picture of \( F \) given in Mandelbrot (1982), which is similar to figure 11. Because \( F \) can be tiled by smaller copies of itself, the probability measure \( \mu \) that we seek to approximate is the uniform one supported on \( F \). As \( F \) is symmetric about its centre, which we represent as the origin \( O \) in the \( z \) plane, we approximate \( \mu \) by the \( \mathcal{P} \)-balanced measure \( \mu(s,a) \) for the i.f.s. \( \{K, w_1(z), w_2(z)\} \), where \( K \subset \mathbb{C} \) is a suitably large disc centred at \( O \), where

\[
w_1(z) = sz + (1-s)a
\]

has fixed point \( a \in K \) and magnification \( s \in \mathbb{C}, |s| < 1 \), and where

\[
w_2(z) = sz - (1-s)a
\]

has fixed point \(-a\) and magnification \( s \). The probabilities are \( p_1 = p_2 = \frac{1}{2} \).

To estimate the moments of \( F \),

\[
M_n = \int_F z^n \, d\mu(z)
\]
we laid a grid of squares \( 0.25 \times 10^{-2} \text{ m} \times 0.25 \times 10^{-2} \text{ m} \), ruled on tracing paper over \( F \), and centred at the centre of \( F \). We attached weight \( 1/N \) to each vertex that lay over \( F \), where \( N = 343 \) was the total number of such vertices. In this way we obtained estimates for \( M_2 \) and \( M_4 \). We also have \( M_0 = 1 \); and by the symmetry about \( O \), \( M_1 = M_3 = 0 \).

Next we computed the two-parameter family of moments
\[
M_n(s, a) = \int_{\mathbb{R}^2} z^n \, d\mu(s, a; z),
\]
following theorem 9. The computation was as follows.
\[
M_2(s, a) = \int z^2 \, d\mu(s, a, z) = \frac{1}{2} \int (w_1(z)^2 + w_2(z)^2) \, d\mu(s, a; z)
\]
\[
= s^2 M_2(s, a) + (1-s)^2 a^2,
\]
whence
\[
M_2(s, a) = \frac{(1-s) a^2}{1+s}.
\]

Similarly,
\[
M_4(s, a) = \frac{1}{2} \int (w_1(z)^4 + w_2(z)^4) \, d\mu(s, a; z)
\]
\[
= s^4 M_4(s, a) + 6s^2(1-s^2) a^2 M_2(s, a) + (1-s)^4 a^4,
\]
whence
\[
M_4(s, a) = \frac{(1+5s^2/1+s^2) (M_2(s, a))^2}{1+s^2}.
\]
Solving the latter equation for \( s^2 \) yields

\[
s^2 = \frac{-1}{5} \left( \frac{(M_4(s,a))^2 - M_4(s,a)}{(M_2(s,a))^2 - M_4(s,a)} \right).
\]

We next set \( M_2(s,a) = M_2 \) (estimate) and \( M_4(s,a) = M_4 \) (estimate), which provides the approximate value

\[
s^2 = 0.064 - i(0.594)
\]

There correspond four possible solution-pairs \((s,a)\); however, these different i.f.s.s all yield exactly the same measure and approximate fractal \( \tilde{F} \) (see Barnsley & Harrington 1984). A picture of \( \tilde{F} \) is shown in figure 12.

Had our moment estimation been exact – for illustrative purposes it was not our aim to be as accurate as possible – we would have obtained

\[
s^2 = 0 - i(0.5),
\]

which yields precisely \( F \).

References


Barnsley, M. F. & Demko, S. G. 1984 Rational approximation of fractals. Springer Verlag


Barnsley, M. F. & Harrington, A. N. 1985 *Physica 14D*. (In the press.)


Brolin, H. 1965 *Ark. Mat. 6*, 103–144.


Gaal, L. 1971 *Classical Galois theory with examples*. Chicago: Markham.


