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Article Information

PhotoJournalTitle: Constructive approximation
PhotoJournalVolume: 2
PhotoJournalIssue: 1
Month:
Year: 1986
Pages: 303-329
Article Author: Barnsley, Michael
Article Title: Fractal functions and interpolation

Citation Information

Cited In: google
Cited Title:
Cited Date:
Cited Volume:
Cited Pages:

OCLC Information

OCLC Number:
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Oa: Journal Title:
Call Number: Math C766 - R05M03S24T12
Location: DEPO

Notes
Fractal Functions and Interpolation

Michael F. Barnsley

Abstract. Let a data set \( \{(x_i, y_i) \in I \times \mathbb{R}; \ i = 0, 1, \ldots, N\} \) be given, where \( I = [x_0, x_N] \subset \mathbb{R} \). We introduce iterated function systems whose attractors \( G \) are graphs of continuous functions \( f: I \rightarrow \mathbb{R} \), which interpolate the data according to \( f(x_i) = y_i \) for \( i \in \{0, 1, \ldots, N\} \). Results are presented on the existence, coding theory, functional equations and moment theory for such fractal interpolation functions. Applications to the approximation of naturally wiggly functions, which may show some kind of geometrical self-similarity under magnification, such as profiles of cloud tops and mountain ranges, are envisaged.

1. Introduction

We introduce some continuous interpolation functions \( f: I \rightarrow \mathbb{R} \), where \( I \) is a real closed interval, which appear ideally suited for the approximation of naturally occurring functions which display some kind of geometrical self-similarity under magnification. The types of functions we want to consider approximating include not only those which describe profiles of mountain ranges, tops of clouds, stalactite hung roofs of caves and horizons over forests, but also wilder functions whose self-similarities may be more of a stochastic origin, such as temperatures in flames as a function of time, electroencephalograph pen traces and even the minute-by-minute stock market index.

We refer to the new interpolation functions as fractal because it can occur that the Hausdorff–Besicovitch dimensions of their graphs are noninteger \([M]\): typically these functions belong to \( \text{Lip}^\delta \) for some \( 0 < \delta < 1 \), which implies they are not differentiable. However, they are not stochastic and are distinct from those treated by Mandelbrot \([M]\) and Fournier, Fussell and Carpenter \([FFC]\), which have been used to produce computer graphics simulations of landscapes. In the present situation the function values are determined uniquely by the given data points together with a number of parameters, and in this way they are analogous to splines and polynomial interpolations. Because of their ability to extrapolate patterns from one scale to all scales, their usage can reflect the expectations of the practitioner concerning the presence of deep-hidden determinism in apparently disorganized data.

Date received: May 23, 1985. Date revised: May 15, 1986. Communicated by Charles A. Micchelli.

AMS classification: 26A, 28A, 41A, 44A, 60J.

Key words and phrases: Interpolation, Fractals, Moment theory.
The fractal interpolation functions introduced here have their roots in the theory of iterated function systems [H, DS, BD 1, BD 2], which is reviewed briefly in Section 2. In Section 3 we introduce the interpolation functions themselves, and pick out several natural classes which are easy to work with. We also comment on the use of condensation sets. In Section 4 we describe the associated coding theory and measure theory. The former yields explicit formulas for calculating function values, while the latter are exploited in Section 5 in the evaluation of integrals: for a wide class of fractal interpolation functions \(f(x)\) we show how to evaluate explicitly moment integrals of the form

\[ \int_I x^n f(x)^m \, dx, \quad n, m = 0, 1, 2, \ldots \]

in terms of the interpolation points and the other parameters which fix \(f(x)\). These numbers can be used in the design of effective approximation schemes. We also use the invariant measures to discover functional equations obeyed by Fourier and Laplace transforms of fractal interpolation functions. In Section 6 we compute upper and lower bounds on the Hausdorff-Besicovitch dimension of fractal interpolation functions and show how these can be methodically sharpened. We also consider connections with some functions introduced by Besicovitch and Ursell.

2. Iterated Function Systems

We set up notation and briefly review ideas fixing the framework of the rest of the paper. The main references are [H, BD 2, DS].

Let \(K\) be a compact metric space with distance function \(d(x, y)\) for \(x, y \in K\). Let \(H\) be the set of all nonempty closed subsets of \(K\). Then \(H\) is a compact metric space with the Hausdorff metric

\[ h(A, B) = \max \{ \sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{x \in B} \inf_{y \in A} d(x, y) \}, \]

which is defined whenever \(A\) and \(B\) are subsets of \(K\).

Let \(w_n : K \to K\) for \(n \in \{1, 2, \ldots, N\}\) be continuous. Then we call \(\{K, w_n : n = 1, 2, \ldots, N\}\) an iterated function system (i.f.s.). Define \(W : H \to H\) by

\[ W(A) = \bigcup_n w_n(A) \quad \text{for} \quad A \in H, \]

where \(w_n(A) = \{w_n(x) : x \in A\}\). Any set \(G \in H\) such that

\[ W(G) = G \]

is called an attractor for the i.f.s.; and the i.f.s. always admits at least one attractor. Indeed, one needs only to start with any \(S \in H\) and form the set of all accumulation points of sequences \(\{S_m\}_{m=1}^{\infty}\), where \(S_m \in W_{\leq m}(S)\) and \(W(W_{\leq m-1}(S))\) for \(m = 1, 2, 3, \ldots\) with \(W^{\infty}(S) = S\).

If, for some \(0 \leq s < 1\) and all \(n \in \{1, 2, \ldots, N\}\),

\[ d(w_n(x), w_n(y)) \leq s \cdot d(x, y) \quad \text{for all} \quad x, y \in K, \]
then the i.f.s. is termed *hyperbolic*. In this case \( W \) is a contraction mapping; it obeys
\[
h(W(A), W(B)) \leq s \cdot h(A, B) \quad \text{for all } A, B \in H;
\]
and consequently admits a unique attractor
\[
G = \lim_{m \to \infty} W^m(S) \quad \text{in } H.
\]

When the attractor \( G \) of an i.f.s. is unique, it can be calculated as follows. Let \( p > 0 \) be a probability vector \( p = (p_1, p_2, \ldots, p_N) \) with each \( p_n > 0 \) and \( \sum p_n = 1 \). Start from \( x_0 \in K \) and define a sequence \( \{x_m\} \) by choosing successively
\[
x_m \in \{w_1(x_{m-1}), w_2(x_{m-1}), \ldots, w_N(x_{m-1})\} \quad \text{for } m \in \{1, 2, 3, \ldots\},
\]
where probability \( p_n \) is attached to the event \( x_m = w_n(x_{m-1}) \). Then, almost surely, \( G \) consists of the set of points \( g \in K \) such that each open neighborhood of \( g \) contains infinitely many \( x_m \)'s. When \( K = [0, 1] \times [0, 1] \subset \mathbb{R}^2 \) for example, such computations are readily effected and displayed on a microcomputer by plotting \( \{x_m: m = 51, 52, \ldots, 5000\} \), say. If the i.f.s. is hyperbolic then the \( x_m \)'s will approach a distribution given by the unique probability measure \( \mu \) on \( K \) which is stationary for the discrete time Markov process where the probability of transfer from \( x \in K \) to a Borel subset \( B \) of \( K \) is
\[
P(x, B) = \sum_p \delta_{w_n(x)}(B).
\]
Here \( \delta_y(B) \) equals one if \( y \in B \) and zero otherwise.

Regardless of whether or not the i.f.s. is hyperbolic, there always exists such a stationary probability measure \( \mu \); and it is referred to as a *p-balanced measure* of the i.f.s. When the attractor \( G \) of the i.f.s. is unique, it follows from [BD 2, Theorem 2] that the support of \( \mu \) is \( G \) independently of \( p > 0 \).

The stationarity of \( \mu \) is equivalent to
\[
(2.1) \quad \int_K f \, d\mu = \int_K \sum_n p_n f \circ w_n \, d\mu \quad \text{for all } f \in C,
\]
where \( C \) denotes the continuous real-valued functions on \( K \). Let \( P(K) \) denote the set of probability measures on \( K \). For given \( \nu_0 \in P(K) \) define \( \nu_m \in P(K) \) by
\[
\nu_m(B) = \sum_p \nu_{m-1}(w_n^{-1}(B)) \quad \text{for } m \in \{1, 2, 3, \ldots\}.
\]
Then for hyperbolic i.f.s. \( \nu_m \) converges to \( \mu \) in the sense that
\[
\lim_{m \to \infty} \int_K f \, d\nu_m = \int_K f \, d\mu \quad \text{for each } f \in C.
\]
These results have applications to computer graphics, see [DS] and [DHN].

Although the above formulation in terms of a Markov process seems very stochastic, a completely deterministic description can be made; see the Remark in Section 4.
3. Fractal Interpolation Functions

Let a set of data points \( \{(x_i, y_i) \in I \times \mathbb{R}: i = 0, 1, \ldots, N\} \) be given, where \( I = [x_0, x_N] \subset \mathbb{R} \) is a closed interval, see Fig. 1. We are concerned with continuous functions \( f: I \to \mathbb{R} \) which interpolate the data according to

\[
f(x_i) = y_i, \quad i = 0, 1, 2, \ldots, N,
\]

see Fig. 2. We focus on the existence, construction, and properties of such functions \( f: I \to \mathbb{R} \) whose graphs

\[
G = \{(x, f(x)) : x \in I\}
\]

are attractors of i.f.s. That is, there is a compact subset \( K \) of \( I \times \mathbb{R} \), and a collection of continuous mappings \( w_n : K \to K \) such that the unique attractor of the i.f.s. \( \{K, w_n : n = 1, 2, \ldots, N\} \) is \( G \). We shall refer to such functions \( f \) as fractal interpolation functions: often the Hausdorff–Besicovitch dimension of \( G \) will be noninteger; \( f \) may then be Hölder continuous but not differentiable.

We will work in the compact metric space \( K = I \times [a, b] \) for some \(-\infty < a < b < \infty\), with the Euclidean metric or the equivalent metric

\[
d((c_1, d_1), (c_2, d_2)) = \max\{|c_1 - c_2|, |d_1 - d_2|\} \quad \text{for} \quad (c_1, d_1), (c_2, d_2) \in K.
\]

Set \( I_n = [x_{n-1}, x_n] \) and let \( L_n : I \to I_n \) for \( n \in \{1, 2, \ldots, N\} \) be contractive homeomorphisms such that

\[
L_n(x_0) = x_{n-1}, \quad L_n(x_N) = x_n,
\]

\[
|L_n(c_1) - L_n(c_2)| \leq l \cdot |c_1 - c_2| \quad \text{whenever} \quad c_1, c_2 \in I,
\]

for some \( 0 \leq l < 1 \). Furthermore, let mappings \( F_n : K \to [a, b] \) be continuous with, for some \( 0 \leq q < 1 \),

\[
F_n(x_0, y_0) = y_{n-1}, \quad F_n(x_N, y_N) = y_n,
\]

\[
|F_n(c, d_1) - F_n(c, d_2)| \leq q \cdot |d_1 - d_2|
\]

for all \( c \in I, d_1 \) and \( d_2 \in [a, b] \), and \( n \in \{1, 2, \ldots, N\} \). Now define functions \( w_n : K \to K \) for \( n \in \{1, 2, \ldots, N\} \) by

\[
w_n(x, y) = (L_n(x), F_n(x, y)).
\]

Then \( \{K, w_n : n = 1, 2, \ldots, N\} \) is an i.f.s. but may not be hyperbolic.

**Theorem 1.** The i.f.s. \( \{K, w_n : n = 1, 2, \ldots, N\} \) defined above has a unique attractor \( G \). \( G \) is the graph of a continuous function \( f : I \to [a, b] \) which obeys

\[
f(x_i) = y_i \quad \text{for} \quad i = 0, 1, 2, \ldots, N.
\]

![Fig. 1](image-url)
Proof. Let $G$ be any attractor of the i.f.s. Let
\[ \hat{I} = \{ x \in I : (x, y) \in G \text{ for some } y \in [a, b] \}. \]
Then, since $G = \bigcup_n w_n(G)$, it follows that $\hat{I} = \bigcup_n L_n(\hat{I})$. But, $\{ I, L_n : n = 1, 2, \ldots, N \}$ is a hyperbolic i.f.s. whose unique attractor is $I$, whence
\[ \hat{I} = I = [x_0, x_N]. \]

Next, we show that $G$ is the graph of a function $f: I \to [a, b]$ by proving that only one $y$-value corresponds to each $x$-value. First we show this for the $x$-values $\{x_0, x_1, \ldots, x_N\}$. Let
\[ S_i = \{ (x, y) \in G : x = x_i \} \quad \text{for } i \in \{0, 1, \ldots, N\}. \]
Since $w_n(S_0) \cap S_0 = \emptyset$ for $n \neq 1$, we must have $w_1(S_0) = S_0$. But $w_1$ is a strict contraction on the compact metric space $S_0$, whence $S_0 = (x_0, y_0)$. Similarly, $S_N = (x_N, y_N)$. For $I \in \{1, 2, \ldots, N - 1\}$ we observe that the only points which can map to $S_i$ are $S_0$ (under $w_{i-1}$) and $S_N$ (under $w_i$), from which $S_i = w_{i-1}(S_0) \cup w_i(S_N) = (x_i, y_i)$.

Consider
\[ \delta = \text{Max}\{ |s - t| : (x, s) \in G, (x, t) \in G, \text{some } x \in I \}. \]
By the compactness of $G$ this maximum is achieved at some pair of points $(\hat{x}, s)$ and $(\hat{x}, t)$ in $G$, with
\[ \delta = |s - t|, \]
and from the last paragraph we can assume $\hat{x} \in (x_{n-1}, x_n)$ for some $n$. But then there exists two points
\[ (L_n^{-1}(\hat{x}), u) \quad \text{and} \quad (L_n^{-1}(\hat{x}), v) \quad \text{both in } G, \]
with
\[ s = F_n(L_n^{-1}(\hat{x}), u) \quad \text{and} \quad t = F_n(L_n^{-1}(\hat{x}), v). \]
Hence
\[ \delta = |s - t| = |F_n(L_n^{-1}(\hat{x}), u) - F_n(L_n^{-1}(\hat{x}), v)| \]
\[ \leq q \cdot |u - v| \leq q \cdot \delta, \]
with \( 0 \leq q < 1 \) whence \( \delta = 0 \). Hence \( G \) is the graph of a function \( f : I \to [a, b] \) which obeys (3.3). \( G \) is unique because the union of two attractors is an attractor.

To prove \( f(x) \) is continuous, let \( C(I) \) denote the Banach space of continuous real-valued functions \( g : I \to \mathbb{R} \), with norm \( |g|_{\infty} = \text{Max}\{ |g(x)| : x \in I \} \). Let \( C_0(I) \subset C(I) \) consist of those \( g \in C(I) \) such that \( g : I \to [a, b] \), and which obey \( g(x_0) = y_0 \) and \( g(x_N) = y_N \). Then \( C_0(I) \) is a complete metric space and the i.f.s. induces a contraction
\[ T : C_0(I) \to C_0(I) \]
defined by
\[ (Tg)(x) = F_n(L_n^{-1}(x), g(L_n^{-1}(x))) \quad \text{when} \quad x \in I_n. \]
We have
\[ |Th - Tg|_{\infty} = \text{Max}\{ |F_n(L_n^{-1}(x), h(L_n^{-1}(x)) - F_n(L_n^{-1}(x), g(L_n^{-1}(x)))| : x \in I_n, \}
\[ n = 1, 2, \ldots, N \}
\[ \leq \text{Max}\{ q \cdot |h(L_n^{-1}(x)) - g(L_n^{-1}(x))| : x \in I_n, n = 1, 2, \ldots, N \}
\[ \leq q \cdot |h - g|_{\infty}. \]
Hence \( T \) has a unique fixed point \( \hat{h} \in C_0(I) \). The graph of \( \hat{h} \) is an attractor of the i.f.s., whence \( \hat{h} = f \), whence \( f \) is continuous.

The above theorem should be compared to a theorem of Hutchinson [H, p. 731] concerning i.f.s. (in our language) whose attractors are continuous parametrized curves passing through a specified set of points. In the present case, by restricting the mappings to be generalized shear transformations, the parametrized curve is discovered to be the graph of a function. Furthermore, the underlying compact metric space is different: here the i.f.s. has \( \hat{G} \) as its attractor in a bigger compact metric space (all closed nonempty subsets of \( I \times [a, b] \) versus all graphs of continuous functions \( g : I \to [a, b] \) such that \( g(x_0) = y_0 \) and \( g(x_N) = y_N \)). In particular, stochastic iteration can be used to calculate \( G \).

**Example 1.** We define linear fractal interpolation functions to be those obtained using affine maps
\[ L_n(x) = x_{n-1} + (x_n - x_{n-1}) \cdot (x - x_0)/(x_N - x_0) = a_n \cdot x + h_n, \]
\[ F_n(x, y) = b_n x + \alpha_n \cdot y + k_n, \]
where the real constants \( b_n \) and \( k_n \), depending on the adjustable real parameter \( \alpha_n \), are chosen to ensure
\[ F_n(x_0, y_0) = y_{n-1} \quad \text{and} \quad F_n(x_N, y_N) = y_N. \]
That is, \( \alpha_n \in (-1, 1) \) is chosen and then
\[
b_n = (y_n - y_{n-1} - \alpha_n \cdot (y_N - y_0))/(x_N - x_0),
\]
\[
k_n = y_{n-1} - \alpha_n \cdot y_0 - b_n x_0 \quad \text{for} \quad n \in \{1, 2, \ldots, N\}.
\]

Then \( w_n \) maps the parallelogram \( P \) onto the parallelogram \( P_n \) in Fig. 3. We have written simple computer programs in compiled Basic on an IBM PC, for plotting the graphs of such linear fractal interpolation functions. One inputs the interpolation points \( (x_i, y_i) \) for \( i = 0, 1, \ldots, N \) and vertical scaling parameters \( \alpha_n \) for \( n = 1, 2, \ldots, N \); whereupon points on the graph of \( f(x) \) are computed using the stochastic iteration method described in Section 2. The results are displayed on a video graphics monitor and may be printed with a graphics printer: typically computation is very fast and a good view of the whole function is quickly obtained. Also, it is straightforward to magnify up or look closer at a given piece of the curve.

In Fig. 4, we use \( x_i = (i/10) \) for \( i = 0, 1, \ldots, 10 \), with \( \alpha_1 = \alpha_2 = \cdots = \alpha_N = \alpha \) taking various values which effectively adjust the dimension of \( \hat{f}(x) \).

In Fig. 5, we show a selection of linear fractal interpolations, this time with different vertical scalings (the \( \alpha_n \)'s).

**Example 2.** We draw attention to the special class where
\[
L_n(x) = a_n \cdot x + h_n,
\]
\[
F_n(x, y) = \alpha_n \cdot y + q_n(x),
\]
with \( F_n(x_0, y_0) = y_{n-1} \) and \( F_n(x_N, y_N) = y_n \). By choosing the functions \( q_n(x) \) appropriately one can fix \emph{a priori} the overall approximate shape of \( f(x) \). When \( \alpha_n = 0 \) we call \( q_n(x) \) a \emph{condensation function} because then the role it plays in the present set-up is the same as that of condensation sets in the general theory of i.f.s.: in particular, we have \( f(x) = q_n(x) \) for \( x \in I_n \); if we start with \( \alpha_n \neq 0 \) and then let \( \alpha_n \to 0 \) then the graph of \( f(x) \) over \( I_n \) "condenses" on that of \( q_n(x) \).
Let us write

\[ q_n(x) = h(L_n(x)) - \alpha_n \cdot b(x), \]

where \( h(x) \) is a continuous real function such that

\[ h(x_i) = y_i \quad \text{for all} \quad i \in \{0, 1, \ldots, N\} \]

and \( b(x) \) is continuous, real and obeys

\[ b(x_0) = y_0 \quad \text{and} \quad b(x_N) = y_N. \]
We call $b(x)$ the base function and $h(x)$ the height function because $f(x)$ is invariant under the following transformation: subtract the base function from $f(x)$, rescale the result vertically by the factor $\alpha_n$, shrink linearly horizontally so that $[x_0, x_N] \rightarrow [x_{n-1}, x_n]$, then add the result to the height function over $I_n$, for each $n$.

In Example 1, $h(x)$ is the piecewise linear interpolation through the data and $b(x)$ is the linear function through $(x_0, y_0)$ and $(x_N, y_N)$. More generally one can...
take $b(x)$ and $h(x)$ to be splines or polynomials. One might choose $b(x)$ to fix the broad outlines of a distant mountain range in profile; or, with a trigonometrical polynomial, one might match the basic rolling shape of a wave train, upon which ripples on all scales are to be eventually superimposed. As the $|\alpha_n|$'s are increased from zero the dimension of the fractal interpolation function increases, see Section 5.

The reason we draw attention to the special class of fractal interpolation functions $f(x)$ which can be generated in this example, is that such functions are explicitly integrable. In Section 4 we show that integrals of the form

$$f_{n,m} = \int_I x^n (f(x))^m \, dx \quad \text{for all} \quad m, n \in \{0, 1, 2, \ldots\}$$

can be evaluated explicitly in terms of integrals of the $q_k(x)$'s.

Thus, we should be able to construct interpolation functions whose overall shapes, dimensions, and various moments can be prescribed.
Example 3. Let $\hat{f} \in C_0(I)$ be given. Let the mappings \( \{w_n: n = 1, 2, \ldots, N\} \) be chosen so that

\[
|\hat{f} - T\hat{f}|_\infty < \varepsilon,
\]

where $T: C_0(I) \to C_0(I)$ is the mapping introduced in the proof of Theorem 1. Let $f$ be the associated fractal interpolation function,

\[
Tf = f, \quad f \in C_0(I).
\]

Then

\[
|\hat{f} - f|_\infty = \lim_{m \to \infty} |\hat{f} - T^m\hat{f}|_\infty \leq \lim_{m \to \infty} \sum_{n=1}^{m} |T^{n-1}\hat{f} - T^n\hat{f}|_\infty
\]

\[
\leq \lim_{m \to \infty} \sum_{n=1}^{\infty} q^{n-1} \cdot |\hat{f} - T\hat{f}|_\infty < \frac{\varepsilon}{1 - q}.
\]
In other words, if we can choose the \( w_n \)'s so that \( T\hat{f} \) is close to \( \hat{f} \), then the corresponding fractal interpolation function will also be close to \( \hat{f} \). This observation is just a slight variant of the Collage theorem [BEHL] which has been successfully applied in computer graphics [DHN].

4. Codes, Formulas, and Measures

We consider an i.f.s. \( \{K, w_n : n = 1, 2, \ldots, N\} \) which gives rise to a fractal interpolation function \( f : I \to \mathbb{R} \), with graph \( G \), as in Theorem 1.

Let \( \Omega \) denote the set of all half-infinite sequences of \( N \) symbols, \( \{1, 2, \ldots, N\} \), so that \( \omega \in \Omega \) if and only if

\[
\omega = (\omega_1, \omega_2, \omega_3, \ldots),
\]

where each \( \omega_i \in \{1, 2, \ldots, N\} \). We reserve the notation \( \omega_i \) for the \( i \)th component of \( \omega \). We give \( \Omega \) the product topology: with the appropriate metric it becomes a compact metric space homeomorphic to the classical Cantor set. Just as in the case of hyperbolic i.f.s., see [H, BD 2], we can construct a continuous onto mapping \( \Phi : \Omega \to G \) by defining

\[
(4.1) \quad \Phi(\omega) = \lim_{m \to \infty} w_{\omega_1} \circ w_{\omega_2} \circ w_{\omega_3} \circ \cdots \circ w_{\omega_m}(w, y).
\]

To see that this limit exists and is independent of \((x, y) \in K\), we consider the hyperbolic i.f.s. \( \{I, L_n : n = 1, 2, \ldots, N\} \), for which we can define a continuous onto mapping \( \Psi : \Omega \to I \), the attractor of the i.f.s., by

\[
\Psi(\omega) = \lim_{m \to \infty} L_{\omega_1} \circ L_{\omega_2} \circ L_{\omega_3} \circ \cdots \circ L_{\omega_m}(x),
\]

where the limit is independent of \( x \in I \). For the moment let \((x, y) \in K\) be fixed and let \( \Phi(\omega) \) denote the set of all accumulation points of the sequence on the right-hand side of (4.1). Then \( \Phi(\omega) \subset G \), and takes the form

\[
\Phi(\omega) = (\Psi(\omega), S),
\]

where \( S \subset [a, b] \) is the set of all limit points of the sequence

\[
\{F_{\omega_1}(\Psi(T\omega)), F_{\omega_2}(\Psi(T^2\omega)), \ldots, F_{\omega_m}(\Psi(T^m\omega), y) \ldots)\}_{m=1}^\infty,
\]

where \( T : \Omega \to \Omega \) is defined by

\[
T(\omega_1, \omega_2, \omega_3, \ldots) = (\omega_2, \omega_3, \omega_4, \ldots).
\]

But by Theorem 1, since \( \Psi(\omega) \) is a single point in \( I \), we must have

\[
\Phi(\omega) = (\Psi(\omega), f(\Psi(\omega))),
\]

whence \( S \) consists of a single point, as desired. Furthermore, since \( \Psi(\omega) \) is independent of \( x \in I \), we must have \( \Phi(\omega) \) is independent of \((x, y) \in K \). Finally, the continuity of \( \Phi : \Omega \to G \) follows from the continuity of \( \Psi : \Omega \to I \) and the continuity of the map \( P : I \to G \) defined by

\[
P(x) = (x, f(x)) \quad \text{for } x \in I.
\]
In particular, we have established the result
\[ f(\Psi(\omega)) = \lim_{m \to \infty} F_{\omega_1}(\Psi(T\omega), F_{\omega_2}(\Psi(T^2\omega), \ldots, F_{\omega_m}(\Psi(T^m\omega), y) \ldots)), \]
for any \( y \in [a, b] \). Observe that \( \Psi(\omega) = \Psi(\tilde{\omega}) \) whenever \( \omega \) and \( \tilde{\omega} \) are equivalent representations of the same number in base \( N \), but that this leads to no ambiguity in using the latter formula to calculate \( f(x) \).

We consider invariant measures next.

**Theorem 2.** Let \( p > 0 \) be a probability vector, and let \( \{ K, w_n: n = 1, 2, \ldots, N \} \) be the i.f.s. of Theorem 1. Then \( \{ K, w_n: n = 1, 2, \ldots, N \} \) admits a unique \( p \)-balanced measure \( \mu \). Let \( \tilde{\mu} \) be the \( p \)-balanced measure for the i.f.s. \( \{ I, L_n: n = 1, 2, \ldots, N \} \), and let \( P: I \to G \) be the homeomorphism \( P(x) = (x, f(x)) \), for \( x \in I \). Then
\[ \tilde{\mu}(\tilde{B}) = \mu(P(\tilde{B})) \]
for all Borel subsets \( \tilde{B} \) of \( I \).

**Proof.** We show that the \( p \)-balanced measure on the graph is simply the lift of the unique invariant measure for the corresponding process on the \( x \)-axis. Let \( \mathcal{P}(G) \) and \( \mathcal{P}(I) \) denote the spaces of probability measures supported on \( G \) and \( I \), respectively, with the appropriate weak *-topologies. We define a homeomorphism \( M: \mathcal{P}(G) \to \mathcal{P}(I) \) by
\[ (M\mu)(\tilde{B}) = \mu(P\tilde{B}) \]
for all \( \mu \in \mathcal{P}(G) \) and all Borel subsets \( \tilde{B} \) of \( I \). The inverse is
\[ (M^{-1}\mu)(\tilde{B}) = \tilde{\mu}(P^{-1}B) \]
for all \( \tilde{\mu} \in \mathcal{P}(I) \) and all Borel subsets \( B \) of \( G \).

Now let \( \mu \) be any \( p \)-balanced measure on \( G \), so that
\[ \mu(B) = \sum_{n=1}^{N} p_n \mu(w_n^{-1}(B)) \]
for all Borel subsets \( B \) of \( G \). Then it is readily verified that
\[ (M\mu)(\tilde{B}) = \sum_{n=1}^{N} p_n \mu(w_n^{-1}(P\tilde{B})) = \sum_{n=1}^{N} p_n \mu(PL_n^{-1}(\tilde{B})) = \sum_{n=1}^{N} p_n (M\mu)(L_n^{-1}(\tilde{B})) \]
which shows \( M\mu = \tilde{\mu} \), the unique \( p \)-balanced measure for the i.f.s. \( \{ I, L_n: n = 1, 2, \ldots, N \} \). Hence \( \mu = M^{-1}\tilde{\mu} \) is also unique.

**Remark.** An i.f.s. \( \{ K, w_n: n = 1, 2, \ldots, N \} \) complete with a \( p \)-balanced measure \( \mu \) supplies a dynamical system in the following way, see also [P]. Consider the space \( M = [0, 1] \times K \) and define a mapping
\[ \xi: M \to M \]
by
\[
\xi(x, y) = \left( \frac{1}{p_i} (x - p_1 - p_2 - \cdots - p_{i-1}), w_i(y) \right) \quad \text{if} \quad i \in \{2, 3, \ldots, N\}
\]
for
\[
x \in (p_1 + \cdots + p_{i-1}, p_1 + p_2 + \cdots + p_i);
\]
and
\[
\xi(x, y) = \left( \frac{1}{p_i} x, w_i(y) \right) \quad \text{if} \quad x \in [0, p_1].
\]
Then the measure \( \omega = \mathcal{L} \times \mu \) on \( M \), where \( \mathcal{L} \) denotes uniform Lebesgue measure on \([0, 1]\), is invariant under \( \xi \). The dynamical system consists of \((M, \xi, \nu)\), and we refer to it as the one associated with the i.f.s. We say that two i.f.s. are isomorphic if their associated dynamical systems are isomorphic. Theorem 2 tells us that the two i.f.s. under discussion there are isomorphic.

The utility of Theorem 2 is that it allows one to change variables in integrals involving \( p \)-balanced measures on the graphs of fractal interpolation functions. Let \( H \in L_1(K, \mu) \). Then
\[
\int_G H(x, y) \, d\mu(x, y) = \int_I H(x, f(x)) \, d\tilde{\mu}(x).
\]
In particular, if we choose the \( L_n \)'s to be affine, as in Examples 1 and 2, and if we choose
\[
p_n = a_n = \frac{x_n - x_{n-1}}{x_N - x_0},
\]
then \( d\tilde{\mu}(x) = dx/(x_N - x_0) \) and
\[
\int_G H(x, y) \, d\mu(x, y) = \frac{1}{x_N - x_0} \int_I H(x, f(x)) \, dx.
\]

5. Integration of Fractal Interpolation Functions

We continue to discuss an i.f.s. \( \{K, w_n: n = 1, 2, \ldots, N\} \) which gives rise to a fractal interpolation function \( f: I \to \mathbb{R} \), as in Theorems 1 and 2.

The stationarity property (2.1) of the associated \( p \)-balanced measure implies that for \( H \in L_1(K, \mu) \),
\[
\int_G H(x, y) \, d\mu(x, y) = \int_G \sum_{n=1}^N p_n H(w_n(x, y)) \, d\mu(x, y)
\]
\[
= \int_G \sum_{n=1}^N p_n H(L_n(x), F_n(x, y)) \, d\mu(x, y).
\]
If we choose the \( L_n(x) \)'s to be affine and such that (4.2) holds, then (4.3) implies the remarkable identity
\[
\int_I H(x, f(x)) \, dx = \sum_{n=1}^N a_n \int_I H(L_n(x), F_n(x, f(x))) \, dx.
\]
Theorem 3. (i) Let \( f(x) \) be a fractal interpolation function generated by the special class of i.f.s. introduced in Example 2. For each \( m \in \{0, 1, 2, \ldots \} \) one can evaluate explicitly the moment integral

\[
 f_m = \int_I x^m f(x) \, dx
\]

in terms of \( f_{m-1}, f_{m-2}, \ldots, f_0, \) the interpolation points, the scaling parameters \( \{\alpha_n: n = 1, 2, \ldots, N\} \) and the moment

\[
 Q_m = \int_I x^m Q(x) \, dx,
\]

where \( Q: I \to \mathbb{R} \) is defined by

\[
 Q(x) = q_n \circ L_n^{-1}(x) \quad \text{for} \quad x \in I_n.
\]

(ii) Let \( f(x) \) be a linear fractal interpolation function as introduced in Example 1. For each \( l \) and \( m \) in \( \{0, 1, 2, \ldots\} \) one can evaluate explicitly the moment integrals

\[
 f_{l,m} = \int_I x^m (f(x))^l \, dx
\]

in terms of \( \{f_{l,0} \cup f_{l,1} : j = 0, 1, \ldots, m-1\} \cup \{f_{l+1,0} : j = 0, 1, \ldots, l+1\} \cup \{f_{l+1,p} : p = 0, 1, \ldots, l-1\} \) the interpolation points and the scaling parameters \( \{\alpha_n: n = 1, 2, \ldots, N\} \).

Proof. (i) Choose \( H(x, y) = x^my \) in (5.1), with \( L_n \) and \( F_n \) given by (3.5), to obtain

\[
 f_m = \sum_{n=1}^{N} \sum_{k=0}^{m} a_n (a_n \cdot x + h_n)^m (a_n \cdot f(x) + q_n(x)) \, dx
\]

\[
 = \sum_{n=1}^{N} \sum_{k=0}^{m} \alpha_n \cdot a_n^{k+1} \cdot h_n^{m-k} \cdot \binom{m}{k} \cdot f_k + Q_m,
\]

where

\[
 Q_m = \sum_{n=1}^{N} a_n \int_I (L_n(x))^m \cdot q_n(x) \, dx
\]

\[
 = \sum_{n=1}^{N} \int_{I_n} \eta^m q_n \circ L_n^{-1}(\eta) \, d\eta = \int_I x^m Q(x) \, dx,
\]

where in the \( n \)th integral we have made the change of variable \( \eta = L_n(x) \). It follows that

\[
 f_m = \left( \sum_{k=0}^{m-1} f_k \cdot \binom{m}{k} \cdot \sum_{n=1}^{N} a_n^{k+1} \cdot \alpha_n \cdot h_n^{m-k} + Q_m \right) / \left( 1 - \sum_{n=1}^{N} a_n^{m+1} \cdot \alpha_n \right).
\]

The denominator is positive because \( \sum_{n=1}^{N} a_n = 1 \) with each \( a_n > 0 \); and each \( \alpha_n \in (-1, 1) \).
(ii) Choose $H(x, y) = x^m y^l$ in (5.1), with $F_n$ and $L_n$ given by (3.4), to obtain

\[ f_{i,m} = \sum_{n=1}^{N} a_n \int_{I} (a_n \cdot x + h_n)^m (\alpha_n \cdot f(x) + b_n x + k_n)^l \, dx \]
\[ = \sum_{p=0}^{l} \int_{I} f(x)^p P_{l,m,p}(x) \, dx, \]

where we define the polynomial $P_{l,m,p}(x)$ and its coefficients, by

\[ P_{l,m,p}(x) = \sum_{n=1}^{N} \binom{l}{p} a_n (b_n \cdot x + k_n)^{l-p} \alpha_n^p (a_n \cdot x + h_n)^m \]
\[ = \sum_{j=0}^{l+m-p} K(l, m, p, j) x^j. \]

Noting that $K(l, m, l, m) = \sum_{n=1}^{N} a_n^{m+1} \cdot \alpha_n \in (-1, 1)$, we obtain

\[ f_{i,m} = \left( \sum_{j=0}^{m-1} f_{i,j} \cdot \binom{m}{j} \sum_{n=1}^{N} a_n^{l+1} \cdot \alpha_n \cdot h_n^{m-j} \right) \left( 1 - \sum_{n=1}^{N} a_n^{m+1} \cdot \alpha_n \right). \]

**Example 4.** Consider the linear fractal interpolation function for the three interpolation points $(0, 0), (1, 1), \text{and} (2, 0)$, with vertical scalings $\alpha_1 = \alpha_2 = \frac{1}{2}$; we find

\[ w_1(x) = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad w_2(x) = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \]

corresponding to $L_1(x) = \frac{1}{2} x, L_2(x) = \frac{1}{2} x + 1, F_1(x, y) = \frac{1}{2} x + \frac{1}{2} y$, and $F_2(x, y) = -\frac{1}{2} x + \frac{1}{2} y + 1$; see Fig. 6.

We readily calculate

\[ f_n = \int_{0}^{x} x^n f(x) \, dx = \frac{1}{2} \int_{0}^{x} (\frac{1}{2} x + \frac{1}{2} f(x))^n \, dx 
\]
\[ + \frac{1}{2} \int_{0}^{x} (\frac{1}{2} x + 1)^n (-\frac{1}{2} x + \frac{1}{2} f(x) + 1) \, dx \]

from which

\[ f_n = \sum_{m=0}^{n-1} \binom{n}{m} \frac{2^{n-m}}{2^{n+2} - 2} f_m + \frac{2^{n+2}}{(n+1)(n+2)}, \]

whence we calculate successively $f_0 = 2, f_1 = 2, f_2 = \frac{53}{21}, \text{etc.}$

Equation (5.1) can provide functional equations satisfied by integral transforms of fractal interpolation functions. These are of interest for a number of reasons. First, the existence of fractal interpolation functions furnishes existence theorems for solutions of the functional equations. Second, these equations will involve renormalization, and thus they supply generalizations and sources of examples...
in statistical physics. We note the successful exploitation by Bellissard [Be] and others of such equations in the construction of limit periodic operators associated with interesting quantum mechanical models, and also by Derrida and others, in connection with partition functions for various Ising models [Der]. Third, such equations may allow one to investigate smoothness properties of fractal interpolations by studying the asymptotics of the Fourier transform, for example, somewhat along the lines in [G].

For the general integral transform

$$\hat{f}(t) = \int_I K(x,t)f(x) \, dx$$

we have from (5.1)

$$\hat{f}(t) = \sum_{n=1}^{N} a_n \int_I K(L_n(x), t) F_n(x, f(x)) \, dx.$$ 

Hence, if $F_n(x, f(x)) = \alpha_n \cdot f(x) + q_n(x)$ as in Example 2, we obtain

$$\hat{f}(t) = \sum_{n=1}^{N} a_n \cdot \alpha_n \cdot \int_I K(L_n(x), t)f(x) \, dx + \hat{Q}(t),$$

where $Q(x) = q_n(L_n^{-1}(x))$ for $x \in I_n$. In particular, when $K(x, t)$ depends either on the product or the difference of $x$ and $t$, the sum on the right-hand side of (5.2) can be re-expressed in terms of $\hat{f}$, as in the following example. Note that these functional equations are related to, but distinct from, ones obeyed by transforms of $p$-balanced measures, cf. [BD 2].
Example 5. The Fourier transform is obtained with \( K(x, t) = e^{ibt} \) in (5.2). We find

\[
\hat{f}(t) = \sum_{n=1}^{N} \alpha_n : a_n \cdot e^{ih_n t} \cdot \hat{f}(a_n t) + \hat{Q}(t).
\]

For the numbers in Example 4, we find \( Q(x) = x \) for \( 0 \leq x \leq 1 \) and \( Q(x) = 2 - x \) for \( 1 \leq x \leq 2 \), so

\[
\hat{f}(t) = \frac{1}{4} \hat{f} \left( \frac{t}{2} \right) + \frac{1}{4} e^{it} \hat{f} \left( \frac{t+1}{2} \right) + \frac{2e^{it} - e^{2it} - 1}{t^2}.
\]

Example 6. The Stieltjes transform is obtained with \( K(x, t) = 1/(t-x) \). We discover, from (5.2),

\[
\hat{f}(t) = \sum_{n=1}^{N} \alpha_n \cdot f \left( \frac{1}{a_n} (t-h_n) \right) + \hat{Q}(t).
\]

For the numbers in Example 4 we find

\[
\hat{f}(t) = \frac{1}{2} \hat{f}(2t) + \frac{1}{2} \hat{f}(2t-2) + t \ln(t/(t-1)) + (t-2) \ln((t-2)/(t-1)).
\]

6. Dimension of Fractal Interpolation Functions

Given a set of points \( A \subset \mathbb{R}^n \), \( 0 < \varepsilon < \infty \), and \( 0 \leq p < \infty \) let

\[
M_p^\varepsilon(A) = \inf \sum_{i=1}^{\infty} \left[ \delta(A_i) \right]^p,
\]

where \( A = \bigcup_i A_i \) is a countable decomposition of \( A \) in subsets of diameter \( \delta(A_i) \) less than \( \varepsilon \). We set \( [\delta(A_i)]^0 = 0 \) when \( A_i \) is empty and \( = 1 \) otherwise. As \( \varepsilon \) increases we have more sets of decompositions to choose from so \( M_p^\varepsilon(A) \) is a monotone decreasing function of \( \varepsilon \). We define

\[
M_p(A) = \lim_{\varepsilon \to 0} M_p^\varepsilon(A) = \sup_{\varepsilon > 0} M_p^\varepsilon(A).
\]

The \( p \)-dimensional measure of \( A \), \( M_p(A) \), is a Borel regular measure, but is not usually bounded on finite sets. For each \( A \) there is a unique real number \( D \), the Hausdorff–Besicovitch dimension of \( A \), such that

\[
M_p(A) = \begin{cases} 
\infty & \text{if } p < D, \\
0 & \text{if } p > D.
\end{cases}
\]

\( M_p(A) \) may take any value in \([0, \infty)\). A reference is [F].

The following theorem develops from [BD 2, Theorem 8], which applies to the dimension of the attractor \( G \) of a hyperbolic i.f.s. such that \( W_i(G) \cap W_j(G) = \emptyset \) for \( i \neq j \). A related result [H, Theorem 5.3(1)] permits the latter condition to be "slightly" violated in that a certain open set condition must be obeyed (this condition is satisfied in the case of fractal interpolation functions), but it applies only when the \( w_n \)'s are similitudes, which is rarely the case here. A mapping \( w_n : \mathbb{R}^2 \to \mathbb{R}^2 \) is a similitude when there is \( s \in \mathbb{R} \) so that \( w_n(x) - w_n(y) = s \cdot (x - y) \) for all \( x, y \in \mathbb{R}^2 \).
Theorem 4. Let \( \{K, w_n: n = 1, 2, \ldots, N\} \) be a hyperbolic i.f.s., which generates a fractal interpolation function \( f \) with graph \( G \) as in Section 3 and such that the mappings \( L_n: I \rightarrow I_n \) are affine with \( L_n(x) = a_n \cdot x + h_n \). Let bounds on the contractivities be expressed by numbers \( 0 < t_n \leq s_n < 1 \) such that for each \( n \)

\[
    t_n d(x, y) \leq d(w_n(x), w_n(y)) \leq s_n d(x, y) \quad \text{for all } x, y \in K,
\]

where \( d(\cdot, \cdot) \) is a metric on \( K \) equivalent to the Euclidean metric. Then the Hausdorff-Besicovitch dimension \( D \) of \( G \) is bounded by

\[
    \min\{2, l\} \leq D \leq u,
\]

where \( l \) and \( u \) are the positive solutions of

\[
    \sum_{n=1}^{N} t_n^l = 1 \quad \text{and} \quad \sum_{n=1}^{N} s_n^u = 1;
\]

and where for the lower bound to hold it must also be true that

\[
    t_1 \cdots t_N \leq (\min\{a_1, a_N\}) \left( \sum_{n=1}^{N} t_n^l \right)^{2/l}.
\]

Proof. Without loss of generality we take \( d(\cdot, \cdot) \) to be the Euclidean metric. In practice it is convenient to take the metric to be the one in (3.1).

To obtain the upper bound let \( G = \bigcup_i A_i \) be a finite decomposition of \( G \) into subsets of diameter \( \varepsilon \), so that \( \sum_i [\delta(A_i)]^p < \infty \) for all \( 0 \leq p \leq \infty \). A new decomposition is provided by

\[
    G = \bigcup_i \bigcup_{j=1}^{N} A_{ij},
\]

where \( A_{ij} = w_j(A_i) \).

Since

\[
    \sum_i \sum_{j=1}^{N} [\delta(A_{ij})]^p \leq \sum_i \sum_{j=1}^{N} |s_j|^p [\delta(A_i)]^p = \sum_{j=1}^{N} |s_j|^p \sum_i [\delta(A_i)]^p,
\]

it follows that whenever \( \sum_{j=1}^{N} |s_j|^p < 1 \) we must have \( M_p^x(G) = 0 \), which implies \( M_p(G) = 0 \) and hence \( p \geq D \).

For the lower bound we use, as in [BD 2, Theorem 8], a theorem of H. P. McKean, see [BI\textit{G}]. In the present setting it states that if \( \mu \) is any probability measure on \( G \), and if \( \beta \in \mathbb{R} \) is such that

\[
    \int_G \int_G \frac{d\mu(x) \ d\mu(y)}{|x-y|^\beta} < \infty
\]

then \( \beta \leq D \).

Let \( \nu_0 \) be the uniform probability measure on \( K \), namely

\[
    \nu_0(B) = \frac{\int_B d\mathcal{L}}{(x_N - x_0)(b-a)}
\]
where $\mathcal{L}$ denotes uniform Lebesgue measure in the plane. Then it is readily verified that

$$J_0^* \int_K \int_K \frac{d\nu_0(x) \, d\nu_0(y)}{|x-y|^\beta} < \infty \quad \text{for all } 0 \leq \beta \leq 2.$$ 

Now define

$$J_{m+1} = \int_K \int_K \frac{d\nu_{m+1}(x) \, d\nu_{m+1}(y)}{|x-y|^\beta} \quad \text{for } m = 0, 1, 2, \ldots,$$

where, as in Section 2,

$$\nu_{m+1}(B) = \sum_{n=1}^N p_n \nu_m(w_n^{-1}(B))$$

for each Borel subset $B$ of $K$. Here $p = (p_1, p_2, \ldots, p_N)$ is a prescribed probability vector; and we know (recall Section 2) that $\nu_m$ converges to the $p$-balanced measure $\mu$, supported on $G$, which is associated with the i.f.s. In particular, the convergence of $\{J_m\}$ to a finite limiting value $J$ as $m \to \infty$ would imply

$$j = \int_G \int_G \frac{d\mu(x) \, d\mu(y)}{|x-y|^\beta} < \infty;$$

so our next aim is to establish values of $\beta$ for which $\{J_m\}$ converges.

Now

$$J_{m+1} = \int_K \int_K |x-y|^{-\beta} \, d\nu_{m+1}(x) \, d\nu_{m+1}(y)$$

$$= \sum_{i=1}^N \sum_{j=1}^N p_i p_j \int_K \int_K |w_i(x) - w_j(y)|^{-\beta} \, d\nu_m(x) \, d\nu_m(y)$$

$$= \sum_{i=1}^N p_i^2 \int_K \int_K |w_i(x) - w_j(y)|^{-\beta} \, d\nu_m(x) \, d\nu_m(y)$$

$$+ \left( \sum_{\substack{i,j=1 \\text{not n.n.} \, i \neq j \, \text{n.n.}}}^N \right) \left( p_i p_j \int_K \int_K |w_i(x) - w_j(y)|^{-\beta} \, d\nu_m(x) \, d\nu_m(y) \right),$$

where "n.n." means "nearest neighbors," namely $i = j \pm 1$. The first term here is dominated by $C(\beta) \cdot J_m$, where

$$C(\beta) = \sum_{i=1}^N p_i^2 \, t_i^{-\beta}.$$ 

The second term is dominated by

$$d(\beta) = \left( \sum_{i,j=1}^N p_i p_j \right) \rho^{-\beta} = \rho^{-\beta},$$

where

$$\rho = \text{Min}\{|x-y| : x \in w_i(K), y \in w_j(K), i \neq j, \text{not n.n.}\}$$

$$\geq \text{Min}\{|x_i - x_j| : i \neq j, \text{not n.n.}\} > 0.$$
To dominate the third term we consider for example

\[ J_m^{1,2} := \int_K \int_K |w_1(x) - w_2(y)|^{-\beta} \, d\nu_m(x) \, d\nu_m(y) \]

\[ \leq \int_I \int_I |L_1(x) - L_2(y)|^{-\beta} \, d\tilde{\nu}_m(x) \, d\tilde{\nu}_m(y) \]

\[ = \int_{I_1} \int_{I_2} |x - y|^{-\beta} \, d\tilde{\nu}_m(L_1^{-1}(x)) \, d\tilde{\nu}_m(L_2^{-1}(y)), \]

where \( \tilde{\nu}_m \) is the measure obtained by projecting \( \nu_m \) onto \( I \) according to

\[ \tilde{\nu}_m(\tilde{B}) = \nu_m(\tilde{B} \times [a, b]) \quad \text{for Borel subsets } \tilde{B} \text{ of } I. \]

It is readily verified, much as in Theorem 2, that

\[ \tilde{\nu}_{m+1}(\tilde{B}) = \sum_{n=1}^N p_n \tilde{\nu}_m(L_n^{-1}(\tilde{B})), \quad m = 0, 1, 2, \ldots \]

from which it follows \( \tilde{\nu}_m \to \tilde{\mu} \), the \( p \)-balanced measure for the i.f.s. \( \{I, L_n: n = 1, 2, \ldots, N\} \).

We will obtain an upper bound for (6.2) by replacing the piecewise constant measures \( \tilde{\nu}_m(L_i^{-1}) \) by point masses, located at the ends of intervals of constancy.

To do this we will use the notation

\[ x_{(i-1),j} = L_i(x_j) \quad \text{for } i = 1, 2, \ldots, N; \quad j = 0, 1, \ldots, N. \]

Then, for example,

\[ x_{i-1} = x_{(i-1),0} < x_{(i-1),1} < x_{(i-1),2} < \cdots < x_{(i-1),(N-1)} < x_{(i-1),N} = x_i. \]

Similarly, we define

\[ x_{(k-1),\ldots,(m-1),j} = L_k(x_{(i-1),\ldots,(m-1),j}) \]

for \( k, l, \ldots, m = 1, 2, \ldots, N \) and \( j = 0, 1, \ldots, N \). For example,

\[ x_{0,(N-1),0} = x_{0,(N-1),0} = L_1(L_N(x_0)) < x_{0,(N-1),1} < x_{0,(N-1),2} < \cdots < x_{0,(N-1),(N-1)} < x_{0,(N-1),N} \]

\[ = L_1(L_N(x_N)) = L_1(x_N) = x_{0,N} = x_1. \]

For \( m = 0 \) in (6.2) we have for \( \beta < 2 \)

\[ J_0^{1,2} = p_1 p_2 S(x_0, x_1, x_2, \beta), \]

where for \( a < b < c \) we define

\[ S(a, b, c, \beta) = \int_a^b \left\{ \int_b^c \frac{1}{(y-x)^\beta} \, dy \right\} \, dx. \]
One readily calculates

\[
S(a, b, c, \beta) = \begin{cases} 
\frac{(c-b)^{2-\beta} + (b-a)^{2-\beta} - (c-a)^{2-\beta}}{(c-a)(b-a)(\beta-1)(\beta-2)} & \text{when } \beta < 2 \text{ with } \beta \neq 1 \\
\frac{(c-a) \ln(c-a) - (c-b) \ln(c-b) - (b-a) \ln(b-a)}{(c-b)(b-a)} & \text{when } \beta = 1.
\end{cases}
\]

For \( m = 1 \) in (6.2) we have

\[
J_1^{1,2} \leq p_1 p_2 \sum_{k=1}^{N-1} \sum_{l=1}^{N-1} \frac{p_k p_{l+1}}{(x_{1,l} - x_{0,k})^\beta} + (p_1 p_2)(p_N p_1)S(x_{0,N-1}, x_1, x_{1,1}, \beta).
\]

To see how this upper bound is obtained consider Figure 7(a) and (b). In Figure 7(a) we show the actual measures \( \tilde{\nu}_1(L_1^{-1}) \) and \( \tilde{\nu}_1(L_2^{-1}) \), and in Figure 7(b) we show the modified measures which majorize the integral.

Fig. 7 (a) The actual measures \( \tilde{\nu}_2(L_1^{-1}) \) and \( \tilde{\nu}_2(L_2^{-1}) \). (b) The measures in (a) modified so that the integral (6.2), with \( m = 1 \), increases.
For $m = 2$ in (6.2) we have, see Figures 8(a) and (b),

$$J_{2}^{1,2} \leq p_1 p_2 \sum_{k=1}^{N-1} \sum_{l=1}^{N-1} \frac{P_k P_{l+1}}{(x_{1,l} - x_{0,k})^\beta}$$

$$+ p_1 p_2 (p_N p_1) \sum_{k=1}^{N-1} \sum_{l=1}^{N-1} \frac{P_k P_{l+1}}{(x_{1,0,l} - x_{0,N-1,k})^\beta}$$

$$+ p_1 p_2 (p_N p_1)^2 S(x_{0,N-1,N-1}, x_1, x_{1,0,1}, \beta).$$

Following the same line in the general case we must obtain

$$J_{m}^{1,2} \leq p_1 p_2 \sum_{k=1}^{N-1} \sum_{l=1}^{N-1} \frac{P_k P_{l+1}}{(x_{1,l} - x_{0,k})^\beta}$$

$$+ p_1 p_2 (p_N p_1) \sum_{k=1}^{N-1} \sum_{l=1}^{N-1} \frac{P_k P_{l+1}}{(x_{1,0,l} - x_{0,N-1,k})^\beta}$$

$$+ p_1 p_2 (p_N p_1)^2 \sum_{k=1}^{N-1} \sum_{l=1}^{N-1} \frac{P_k P_{l+1}}{(x_{1,0,0,l} - x_{0,N-1,N-1,k})^\beta} + \ldots$$

$$+ p_1 p_2 (p_N p_1)^{m-1} \sum_{k=1}^{N-1} \sum_{l=1}^{N-1} \frac{P_k P_{l}}{(x_{1,0,\ldots,0,l} - x_{0,N-1,\ldots,N-1,k})^\beta}$$

$$+ p_1 p_2 (p_N p_1)^m S(x_{0,N-1,\ldots,N-1}, x_1, x_{1,0,\ldots,0,1}, \beta).$$

Hence

$$J_{m}^{1,2} \leq \frac{p_1 p_2}{(1 - p_N p_1 / a^\beta)} \sum_{k=1}^{N-1} \sum_{l=1}^{N-1} \frac{P_k P_{l+1}}{(x_{1,l} - x_{0,k})^\beta} + p_1 p_2 \left(\frac{p_N p_1}{a^\beta}\right)^m S(x_0, x_1, x_2, \beta),$$

where

$$a = \min\{a_1, a_N\}.$$

We assume $p_1 p_N < a^\beta$ so the last term goes to zero as $m \to \infty$. For example, if $p_1 = p_N = 1/N$ and $a = 1/N$ then this requirement is equivalent to $N^\beta < N^2$, namely $\beta < 2$.

Putting results together, we have that the last term in (6.1) is dominated by

$$\sum_{i,j} J_{m}^{i,j} \leq \frac{2}{(1 - p_N p_1 / a^\beta)} \sum_{i=0}^{N-2} \sum_{j=1}^{N-1} \frac{P_{i+1} P_{i+2}}{(x_{i+1,l} - x_{i,k})^\beta}$$

$$+ \left(\frac{p_N p_1}{a^\beta}\right)^m \sum_{i=0}^{N-2} p_{i+1} p_{i+2} S(x_i, x_{i+1}, x_{i+2}, \beta)$$

$$\leq e(\beta) < \infty \quad \text{for all} \quad \beta > 2,$$
where \( e(\beta) \) is independent of \( m \). It follows that

\[
J_{m+1} \leq C(\beta) \cdot J_m + f(\beta) \quad \text{for} \quad m = 0, 1, 2, \ldots,
\]

where \( f(\beta) = d(\beta) + e(\beta) < \infty \) is a positive constant independent of \( m \), for \( \beta < 2 \), and

\[
C(\beta) = \sum_{n=1}^{N} p_n^2 t_n^{-\beta},
\]

together with the condition \( p_N p_1 < a^{\beta} \). We anticipate that we will finally choose

\[
(6.4) \quad p_n = \frac{t_n^\beta}{\sum_{n=1}^{N} t_n^\beta} \quad \text{for} \quad n = 1, 2, \ldots, N;
\]

and thus we impose the requirement

\[
(6.5) \quad t_1 t_N < \text{Min}\{a_1, a_N\} \left( \sum_{n=1}^{N} t_n^\beta \right)^{2/\beta}.
\]
From (6.3) we find inductively

\[ J_{m+1} = (C(\beta))^{m+1} J_0 + \sum_{j=0}^{m} (C(\beta))^j f(\beta) \quad \text{for} \quad m = 0, 1, 2, \ldots \]

It follows that if \( 0 < C(\beta) < 1 \) then \( \{ J_m \} \) is uniformly bounded, so

\[ \int_G \int_G |x-y|^{-\beta} \, d\mu(x) \, d\mu(y) < \infty. \]

Hence

\[ D = \text{Sup} \{ \beta < 2: \sum_{n=1}^{N} p_n^2 t_n^{-\beta} < 1, p_1 p_N < a^\beta \}, \]

from which we get

\[ D \geq \text{Min} \{2, \hat{\beta} \}, \]

where \( \hat{\beta} \) is the positive solution of

\[ \sum_{n=1}^{N} p_n^2 t_n^{-\beta} = 1, \]

where we assume \( p_1 p_N \leq a^\beta \). Hence choosing the \( p_n \) as in (6.4), and assuming (6.5) but with possible equality, we find

\[ D \geq \text{Min} \{2, l \}, \]

where \( l \) is the positive solution of

\[ \sum_{n=1}^{N} t_n^l = 1. \]

In practice this theorem may not give good numbers in general, especially when the \( s_n \)'s and \( t_n \)'s are evaluated with respect to the Euclidean metric. However, the bounds can often be tightened by iterating the i.f.s.: if \( f \) corresponds to \( \{ K, w_n : n = 1, 2, \ldots, N \} \) then it also corresponds to \( \{ K, w_n \circ w_m : n, m = 1, 2, \ldots, N \} \), and in the obvious notation, for affine maps, we have

\[ t_n \cdot t_m \leq t_{n,m} \quad \text{and} \quad s_{n,m} \leq s_n \cdot s_m. \]

When these inequalities are strict, as they typically will be, we observe that if

\[ \sum_{n=1}^{N} t_n^l = 1, \]

then

\[ \sum_{n=1}^{N} \sum_{m=1}^{N} t_n^l t_m^l = 1, \]

which implies

\[ \sum_{n=1}^{N} \sum_{m=1}^{N} t_{n,m}^l > 1; \]
so the solution $\tilde{t}$ of
\[ \sum_{n=1}^{N} \sum_{m=1}^{N} t_{n,m} = 1 \]
must obey
\[ I < \tilde{t}. \]
Similarly, the upper bound may be improved.

7. Hölder Continuity and Some Functions of Besicovitch

Besicovitch and Ursell [BU] prove the following. Let $y = f(x)$ denote a curve in $\mathbb{R}^2$, with $f: I \rightarrow \mathbb{R}^2$ belonging to the Lipschitz $\delta$-class ($\text{Lip}^\delta$), with $D$ the Hausdorff-Besicovitch dimension of its graph $G$. Then
\[ 1 \leq D \leq 2 - \delta. \]
The bounds are sharp: a curve of class $\text{Lip}^\delta$ may have $D$ anywhere in the above range.

$f(x)$ with domain $I$ belongs to $\text{Lip}^\delta$ if there is a constant $C$ so that for any $x \in I^\delta$ there corresponds an interval $(x-h, x+h)$ such that, for any $y$ in this interval
\[ |f(x) - f(y)| < C \cdot |x - y|^\delta. \]

In proving sharpness Besicovitch considers special functions $f(x)$ of the following type. Write $\varphi(x)$ for the function equal to $2x$ in $0 \leq x \leq \frac{1}{2}$ and defined elsewhere by the relations
\[ \varphi(x) = \Phi(-x) = \Phi(x+1). \]
Then define
\[ f(x) = \sum_{n=0}^{\infty} a_n \Phi(b_n x), \quad x \in [0, 1], \]
where
\[ a_n = b_n^{-\delta}, \quad 0 < \delta < 1. \]
If $b_{n+1} > B \cdot b_n$ where $B > 1$ then $f(x)$ is of class $\text{Lip}^\delta$ and not of any higher Lipschitz class.

An example of such a function is
\[ f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n \delta} \Phi(2^n x). \]
On the interval $[0, 1]$ this is exactly the linear fractal interpolation function (Example 1) which passes through $(0, 0), (\frac{1}{2}, \frac{1}{2})$ and $(1, 0)$, with vertical scalings $\alpha_1 = \alpha_2 = 1/2^\delta$.

Acknowledgments. Helpful discussions related to this work were had with Stephen Demko and Douglas Hardin. This research was supported in part by National Science Foundation Grant DMS-8401609
References


M. F. Barnsley
School of Mathematics
Georgia Institute of Technology
Atlanta
Georgia 30332
U.S.A.